THE INVERSE OF A SQUARE MATRIX **SECTION 8.3** **NOTES**

The Inverse of a Matrix:

$$ax = b \qquad (a^{-1}\cdot a)x = a^{-1}b \qquad 1\cdot x = a^{-1}b \qquad x = a^{-1}b$$

$$AX = B \qquad A^{-1}AX = A^{-1}\cdot B \qquad IX = A^{-1}\cdot B \qquad X = A^{-1}\cdot B$$

Definition: Let A be an n x n matrix and let I_n be the n x n identity matrix. If there exists a matrix A^{-1} such that

square matrix

$$AA^{-1} = I_n = A^{-1}A$$

then A^{-1} is called the inverse of A. The symbol A^{-1} is read "A "inverse."

* only square matrices can have an identity matrix

To show that A and B are inverses, show that $AB = I = BA$.

Example:

If inverses, product is the identity matrix.

1. Show that B is the inverse of A.

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}$$

AB

$$\begin{bmatrix} (2)(-1)+(-1)(-3) & (2)(-1)+(-1)(-2) \\ (-3)(-1)+(1)(-3) & (-3)(-1)+(1)(-2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I

$$\begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$$

BA

$$\begin{bmatrix} -2+3 & 1+-1 \\ -6+6 & 3+-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I

If a matrix A has an inverse, then A is called invertible (or nonsingular); otherwise A is called **singular**. A non-square matrix cannot have an inverse.

Finding the Inverse of a Matrix:

Example: Find the inverse.

you want to get the identity matrix on the left. A^{-1} will be on the right.

2. $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right]$$

$R_1 + R_2 \to R_2$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$2R_2 + R_1 \to R_1$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ \hline 1 & 0 & 3 & 2 \end{array}$$

$$A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

3. $A = \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$-R_2 \to R_2$

$-R_1 + R_3 \to R_3$

$$\begin{array}{cccccc} -1 & 2 & 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$2R_2 + R_1 \to R_1$

$$\begin{array}{cccccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 2 & -4 & 0 & -2 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cccccc} 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 2 & -2 & 0 & 2 \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 0 & 5 & -5 & 0 & 5 \end{array}$$

$5R_3 + R_1 \to R_1$

$2R_3 + R_2 \to R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -2 & 5 \\ 0 & 1 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -4 & -2 & 5 \\ -2 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Determinant:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$|A| = \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

ad - cb
or ad - bc

$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ $|A| = 3 - 2 = 1$

$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

The determinant can be used to find the inverse of a 2 x 2 matrix.

$$\frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example:

4. If possible, find the inverse of $A = \begin{bmatrix} 5 & -1 \\ 3 & 4 \end{bmatrix}$.

$|A| = (5)(4) - (3)(-1)$
$= 20 + 3$
$= 23$

$A^{-1} = \frac{1}{23}\begin{bmatrix} 4 & 1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{4}{23} & \frac{1}{23} \\ \frac{-3}{23} & \frac{5}{23} \end{bmatrix}$

A System of Equations with a Unique Solution

If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by $X = A^{-1}B$.

Example:

5. Use an inverse matrix to solve the system $\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1 \\ 2x + 4y + z = -2 \end{cases}$.

$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$

$A \cdot X = B$

$X = A^{-1} \cdot B$

$X = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ $(2, -1, -2)$

2^{nd} Matrix $\boxed{x^{-1}}$ Edit to enter the matrix
row x column
entries

$A^{-1}B$
2^{nd} Matrix $\boxed{x^{-1}}$ Name choose $\boxed{1}$: A (to get A)
$\boxed{x^{-1}}$ (to get A^{-1})

2^{nd} Matrix $\boxed{x^{-1}}$ Name choose $\boxed{2}$: B

Should have $A^{-1}B$

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