Representation of Matrices: Capitol letters A, B, C, etc.
$[a_{ij}]$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 0 & -6 \end{bmatrix}$$

Equal Matrices: Same order (dimensions)
corresponding entries are equal

Example:

1. Solve for a_{11}, a_{12}, a_{21}, and a_{22} in the following matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -2 & 4 \end{bmatrix}$$

$a_{11} = 6$ $a_{21} = -2$

$a_{12} = 3$ $a_{22} = 4$

$$\begin{bmatrix} 3 & -4 \\ 5 & x \end{bmatrix} = \begin{bmatrix} y+1 & -4 \\ z & 7 \end{bmatrix}$$

$y+1=3$
$y=2$
$z=5$
$x=7$

Matrix Addition, Subtraction, and Scalar Multiplication:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$A+B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$

Dimensions (order) must be the same to add and subtract

Examples:

2. Evaluate the expression.

$$\begin{bmatrix} 4 & -1 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 4+2 & -1+-1 \\ 2+0 & -3+6 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & -2 \\ 2 & 3 \end{bmatrix}}$$

Scalar multiplication

$$3\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

3. For the following matrices, find (a) $3A$, (b) $-B$, and (c) $3A - B$. → $3A + -B$

$$A = \begin{bmatrix} 4 & -1 \\ 0 & 4 \\ -3 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 4 \\ -1 & 3 \\ 1 & 7 \end{bmatrix}$$

a) $3A = \begin{bmatrix} 12 & -3 \\ 0 & 12 \\ -9 & 24 \end{bmatrix}$

b) $-B = \begin{bmatrix} 0 & -4 \\ 1 & -3 \\ -1 & -7 \end{bmatrix}$

c) $\begin{bmatrix} 12 & -7 \\ 1 & 9 \\ -10 & 17 \end{bmatrix}$

Properties of Matrix Addition and Scalar Multiplication

Let A, B, and C be $m \times n$ matrices and let c and d be scalars. — constant

1. $A+B=B+A$ Commutative property of matrix addition
2. $A+(B+C)=(A+B)+C$ Associative " " " "
3. $(cd)A = c(dA)$ " " " scalar multiplication
4. $IA = A$ Scalar identity property
5. $c(A+B)=cA+cB$ Distributive property
6. $(c+d)A = cA+dA$ " "

Examples: Evaluate the expressions.

4. $\begin{bmatrix} 3 & -8 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 6 & -5 \end{bmatrix} + \begin{bmatrix} 0 & 7 \\ 4 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 10 & -4 \end{bmatrix}$$

5. $2\left(\begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} -4 & 0 \\ -3 & 1 \end{bmatrix}\right)$

$$2\begin{bmatrix} -3 & 3 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ -10 & 6 \end{bmatrix}$$

Zero matrix: All entries are zero. $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad A+0=A$

Additive Identity: zero matrix

$$A+0=A$$

Example:

6. Solve for X in the equation $2X - A = B$, where $A = \begin{bmatrix} 6 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}$.

$$2X - A = B$$
$$+A \quad +A$$
$$\tfrac{1}{2}\cdot 2X = (B+A)\cdot\tfrac{1}{2}$$
$$X = \tfrac{1}{2}(B+A)$$

$$X = \tfrac{1}{2}\begin{bmatrix} 10 & 0 \\ -2 & 8 \end{bmatrix}$$

$$\boxed{X = \begin{bmatrix} 5 & 0 \\ -1 & 4 \end{bmatrix}}$$

Matrix Multiplication

$$\underbrace{\underset{m\times n}{A} \cdot \underset{n \times p}{B}}_{\text{same}}$$

new dimensions: $m \times p$

columns in 1st matrix must = # rows in 2nd matrix

Examples:

7. Find the product AB using $A = \begin{bmatrix} -1 & 4 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 0 & 7 \end{bmatrix}$.

$A \cdot B$

$3\times 2 \quad 2\times 2$

new: 3×2

$$\begin{bmatrix} \overset{R_1C_1}{(-1)(1)+(4)(0)} & \overset{R_1C_2}{(-1)(-2)+(4)(7)} \\ -1 + 0 & 2 + 28 \\ \overset{R_2C_1}{(2)(1)+(0)(0)} & \overset{R_2C_2}{(2)(-2)+(0)(7)} \\ 2 + 0 & -4 + 0 \\ \overset{R_3C_1}{(1)(1)+(2)(0)} & \overset{R_3C_2}{(1)(-2)+(2)(7)} \\ 1 + 0 & -2 + 14 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 30 \\ 2 & -4 \\ 1 & 12 \end{bmatrix}}$$

8. Find the product AB using $A=\begin{bmatrix}0 & 4 & -3\\ 2 & 1 & 7\\ 3 & -2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-2 & 0\\ 0 & -4\\ 1 & 2\end{bmatrix}$.

$\underbrace{3\times 3 \quad 3\times 2}$

$$\begin{bmatrix}\underset{0\,+\,0\,+\,-3}{(0)(-2)+(4)(0)+(-3)(1)} & \underset{0\quad -16\quad -6}{(0)(0)+(4)(-4)+(-3)(2)}\\ \underset{-4\quad 0\quad 7}{(2)(-2)+(1)(0)+(7)(1)} & \underset{0\quad -4\quad 14}{(2)(0)+(1)(-4)+(7)(2)}\\ \underset{-6\quad 0\quad 1}{(3)(-2)+(-2)(0)+(1)(1)} & \underset{0\quad 8\quad 2}{(3)(0)+(-2)(-4)+(1)(2)}\end{bmatrix} = \boxed{\begin{bmatrix}-3 & -22\\ 3 & 10\\ -5 & 10\end{bmatrix}}$$

9. Find, if possible, the product AB using $A=\begin{bmatrix}3 & 1 & 2\\ 7 & 0 & -2\end{bmatrix}$ and $B=\begin{bmatrix}6 & 4\\ 2 & -1\end{bmatrix}$.

$2\times\underbrace{3 \qquad 2}_{\text{not same}}\times 2$

NOT POSSIBLE

10. Find AB and BA using $A=[3 \quad -1]$ and $B=\begin{bmatrix}1\\ -3\end{bmatrix}$.

$B=\begin{bmatrix}1\\ -3\end{bmatrix} \quad A=\begin{bmatrix}3 & -1\end{bmatrix}$

$AB: \; 1\times\underbrace{2 \quad 2}\times 1$

1×1

$$\begin{bmatrix}\underset{3}{(3)(1)}+\underset{3}{(-1)(-3)}\end{bmatrix}$$

$AB=\boxed{[6]}$

$BA: \; 2\times\underbrace{1 \quad 1}\times 2$

2×2

$$BA=\begin{bmatrix}(1)(3) & (1)(-1)\\ (-3)(3) & (-3)(-1)\end{bmatrix}=\boxed{\begin{bmatrix}3 & -1\\ -9 & 3\end{bmatrix}}$$

multiplication is NOT commutative

11. Find A^2, where $A=\begin{bmatrix}2 & 1\\ 3 & -2\end{bmatrix}$.

$$A^2=\begin{bmatrix}2 & 1\\ 3 & -2\end{bmatrix}\cdot\begin{bmatrix}2 & 1\\ 3 & -2\end{bmatrix}$$

$$=\begin{bmatrix}4+3 & 2+-2\\ 6+-6 & 3+4\end{bmatrix} = \boxed{\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}}$$

12. For the system of linear equations, (a) write the system as a matrix equation, $AX = B$, and (b) use Gauss-Jordan elimination on $[A \vdots B]$ to solve for the matrix X.

$$\begin{cases} -2x_1 - 3x_2 = -4 \\ 6x_1 + x_2 = -36 \end{cases}$$

a) $\begin{bmatrix} -2 & -3 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -36 \end{bmatrix}$

coefficient variable constant

b) $\left[\begin{array}{cc|c} -2 & -3 & -4 \\ 6 & 1 & -36 \end{array}\right]$

$-\frac{1}{2}R_1 \to R_1$

$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & 2 \\ 6 & 1 & -36 \end{array}\right]$

$$\begin{array}{ccc} -6 & -9 & -12 \\ 6 & 1 & -36 \\ \hline 0 & -8 & -48 \end{array}$$

$-6R_1 + R_2 \to R_2$

$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & 2 \\ 0 & -8 & -48 \end{array}\right]$

$-\frac{1}{8}R_2 \to R_2$

$\left[\begin{array}{cc|c} 1 & \frac{3}{2} & 2 \\ 0 & 1 & 6 \end{array}\right]$

$$\begin{array}{ccc} 1 & \frac{3}{2} & 2 \\ 0 & -\frac{3}{2} & -9 \\ \hline 1 & 0 & -7 \end{array}$$

$-\frac{3}{2}R_2 + R_1 \to R_1$

$\left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & 6 \end{array}\right]$

$x_1 = -7$

$x_2 = 6$

$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 6 \end{bmatrix}}$

X

$AX = B$

$\begin{bmatrix} -2 & -3 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} -7 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ -36 \end{bmatrix}$

p. 561: 5, 7, 11, 15, 17, 19, 27, 31, 33, 35, 37, 41, 49, 57, 61