MATRICES AND SYSTEMS OF EQUATIONS **SECTION 8.1** **NOTES**

Matrix: A rectangular array of real numbers. The plural of matrix is matrices.

$3x+4y=5$
$x-2y=1$

$Ax+By=C$

$$\left[\begin{array}{cc|c} 3 & 4 & 5 \\ 1 & -2 & 1 \end{array}\right]$$

$m \times n$

$$\begin{array}{c|c} & \begin{array}{ccccc} \text{col.1} & \text{col.2} & \text{col.3} & & \text{col. } n \end{array} \\ \begin{array}{c} \text{row 1} \\ \text{row 2} \\ \vdots \\ \\ \text{row } m \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

main diagonal
$a_{11}, a_{22}, a_{33} \ldots$

(Dimensions)
Order: 2×3
$m \times n$
row x column

row matrix: one row $\begin{bmatrix} 2 & \pi & -1 & 7 \end{bmatrix}$

column matrix: one column $\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$

square matrix: $m \times m$ (or $n \times n$) $\begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix}$

Example:

1. Determine the order (dimensions) of the matrix $\begin{bmatrix} 14 & 7 & 10 \\ -2 & -3 & -8 \end{bmatrix}$.

$\boxed{2 \times 3}$

Augmented Matrix vs. **Coefficient Matrix**

System: $\begin{cases} x-4y+3z=5 \\ -x+3y-z=-3 \\ 2x \quad\; -4z=6 \end{cases}$

augmented
$$\left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right]$$

coefficient
$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

Example:
2. Write the augmented matrix for the system of linear equations.

$\begin{cases} x+y+z=2 \\ 2x-y+3z=-1 \\ -x+2y-z=4 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & 3 & -1 \\ -1 & 2 & -1 & 4 \end{array}\right]$$

Elementary Row Operations

R : row

Operation	Notation
1. Interchange two rows.	$R_a \leftrightarrow R_b$
2. Multiply a row by a nonzero constant.	$cR_a \ (c \neq 0)$
3. Add a multiple of a row to another row.	$cR_a + R_b$

Example:

3. Identify the elementary row operation being performed to obtain the new row-equivalent matrix.

Original Matrix **New Row-Equivalent Matrix**

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 7 \\ 2 & -6 & 14 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & -6 & 14 \end{bmatrix} \qquad (-3)R_1 + R_2 \rightarrow R_2$$

Gaussian Elimination with Back-Substitution

zeros below main diagonal

Use row operations with back-substitution to solve a system of linear equations using a matrix.

Example:

4. Solve using an augmented matrix.

$$\begin{cases} 2x + y - z = -3 \\ 4x - 2y + 2z = -2 \\ -6x + 5y + 4z = 10 \end{cases}$$

$\boxed{(-1, 0, 1)}$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & -3 \\ 4 & -2 & 2 & -2 \\ -6 & 5 & 4 & 10 \end{array}\right]$$

$-2R_1 + R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & -3 \\ 0 & -4 & 4 & 4 \\ -6 & 5 & 4 & 10 \end{array}\right]$$

$3R_1 + R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & -3 \\ 0 & -4 & 4 & 4 \\ 0 & 8 & 1 & 1 \end{array}\right]$$

$2R_2 + R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & -3 \\ 0 & -4 & 4 & 4 \\ 0 & 0 & 9 & 9 \end{array}\right]$$

$9z = 9$

$z = 1$

$-4y + 4(1) = 4$

$\quad -4 \quad -4$

$\frac{-4y}{-4} = \frac{0}{-4}$

$y = 0$

$2x + (0) - 1(1) = -3$

$2x - 1 = -3$

$\quad +1 \quad +1$

$\frac{2x}{2} = \frac{-2}{2}$

$x = -1$

$$\begin{array}{cccc} -4 & -2 & 2 & 6 \\ 4 & -2 & 2 & -2 \\ \hline 0 & -4 & 4 & 4 \end{array}$$

$$\begin{array}{cccc} 6 & 3 & -3 & -9 \\ -6 & 5 & 4 & 10 \\ \hline 0 & 8 & 1 & 1 \end{array}$$

$$\begin{array}{cccc} 0 & -8 & 8 & 8 \\ 0 & 8 & 1 & 1 \\ \hline 0 & 0 & 9 & 9 \end{array}$$

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties. (Echelon refers to the stair-step pattern.)

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in *row-echelon form* is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

Examples:

5. Determine whether the matrix is in row-echelon form. If it is, determine whether it is in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 11 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Zeros above and below the ones so this is in <u>reduced row-echelon</u> form

6. Use Gaussian Elimination to solve the system $\begin{cases} -3x+5y+3z=-19 \\ 3x+4y+4z=8 \\ 4x-8y-6z=26 \end{cases}$. w/ row-eschelon

$$\left[\begin{array}{ccc|c} -3 & 5 & 3 & -19 \\ 3 & 4 & 4 & 8 \\ 4 & -8 & -6 & 26 \end{array}\right]$$

$\boxed{(4,-2,1)}$

$R_1+R_3 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & -3 & -3 & 7 \\ 3 & 4 & 4 & 8 \\ 4 & -8 & -6 & 26 \end{array}\right]$$

$-3R_1+R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -3 & -3 & 7 \\ 0 & 13 & 13 & -13 \\ 4 & -8 & -6 & 26 \end{array}\right]$$

$-4R_1+R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -3 & -3 & 7 \\ 0 & 13 & 13 & -13 \\ 0 & 4 & 6 & -2 \end{array}\right]$$

$\frac{1}{13}R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -3 & -3 & 7 \\ 0 & 1 & 1 & -1 \\ 0 & 4 & 6 & -2 \end{array}\right]$$

$-4R_2+R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -3 & -3 & 7 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$\frac{1}{2}R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -3 & -3 & 7 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{cccc} -3 & 9 & 9 & -21 \\ 3 & 4 & 4 & 8 \\ \hline 0 & 13 & 13 & -13 \end{array}$$

$$\begin{array}{cccc} -4 & 12 & 12 & -28 \\ 4 & -8 & -6 & 26 \\ \hline 0 & 4 & 6 & -2 \end{array}$$

$$\begin{array}{cccc} 0 & -4 & -4 & 4 \\ 0 & 4 & 6 & -2 \\ \hline 0 & 0 & 2 & 2 \end{array}$$

$x-3y-3z=7$
$y+z=-1$
$z=1$

$y+(1)=-1$
$\quad -1 \quad -1$
$y=-2$

$x-3(-2)-3(1)=7$
$x+6-3=7$
$x+3=7$
$\quad -3 \quad -3$
$x=4$

p. 547: 12, 16, 21, 25, 30, 31, 45, 46, 47, 57, 60, 65, 70, 75, 81

7. Use Gaussian Elimination to solve the system $\begin{cases} x+y+z=1 \\ x+2y+2z=2 \\ x-y-z=1 \end{cases}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & -1 & -1 & 1 \end{array}\right]$$

$-R_1+R_2 \to R_2$
$-R_1+R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \end{array}\right]$$

$2R_2+R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$0 \neq 2$

$\boxed{\text{No solution}}$

-1	-1	-1	-1
1	2	2	2
0	1	1	1

-1	-1	-1	-1
1	-1	-1	1
0	-2	-2	0

0	2	2	2
0	-2	-2	0
0	0	0	2

8. Use Gauss-Jordan Elimination to solve the system $\begin{cases} -3x+7y+2z=1 \\ -5x+3y-5z=-8 \\ 2x-2y-3z=15 \end{cases}$.

(zeros above and below)

$$\left[\begin{array}{ccc|c} -3 & 7 & 2 & 1 \\ -5 & 3 & -5 & -8 \\ 2 & -2 & -3 & 15 \end{array}\right]$$

$R_1+R_3 \to R_1$

$$\left[\begin{array}{ccc|c} -1 & 5 & -1 & 16 \\ -5 & 3 & -5 & -8 \\ 2 & -2 & -3 & 15 \end{array}\right]$$

$-1 \cdot R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & -16 \\ -5 & 3 & -5 & -8 \\ 2 & -2 & -3 & 15 \end{array}\right]$$

$5R_1+R_2 \to R_2$
$-2R_1+R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & -16 \\ 0 & -22 & 0 & -88 \\ 0 & 8 & -5 & 47 \end{array}\right]$$

$-\frac{1}{22}R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & -16 \\ 0 & 1 & 0 & 4 \\ 0 & 8 & -5 & 47 \end{array}\right]$$

$-8R_2+R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & -16 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -5 & 15 \end{array}\right]$$

$-\frac{1}{5}R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & -16 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$5R_2+R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$-R_3+R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$\boxed{(7, 4, -3)}$

5	-25	5	-80
-5	3	-5	-8
0	-22	0	-88

-2	10	-2	32
2	-2	-3	15
0	8	-5	47

0	-8	0	-32
0	8	-5	47
0	0	-5	15

9. Solve the system $\begin{cases} 2x-6y+6z=46 \\ 2x-3y \qquad =31 \end{cases}$.

$$\left[\begin{array}{ccc|c} 2 & -6 & 6 & 46 \\ 2 & -3 & 0 & 31 \end{array}\right]$$

$\frac{1}{2}R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & -3 & 3 & 23 \\ 2 & -3 & 0 & 31 \end{array}\right]$$

$-2R_1+R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -3 & 3 & 23 \\ 0 & 3 & -6 & -15 \end{array}\right]$$

$\frac{1}{3}R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -3 & 3 & 23 \\ 0 & 1 & -2 & -5 \end{array}\right]$$

$3R_2+R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & -2 & -5 \end{array}\right]$$

x + y are dependent on z

$z \to a$ (any number)

$y-2a=-5$ $\quad x-3a=8$
$y=2a-5$ $\quad x=3a+8$

$\boxed{(3a+8, 2a-5, a)}$ infinitely many solutions

-2	6	-6	-46
2	-3	0	31
0	3	-6	-15

0	3	-6	-15
1	-3	3	23
1	0	-3	8