Row-Echelon Form and Back-Substitution

Row-echelon form has a "stair-step" pattern. It is easier to solve the system in row-echelon form using back-substitution.

Example:
1. Solve the system of linear equations.

$$\begin{cases} 2x - y + 5z = 22 \\ \quad\; y + 3z = 6 \\ \qquad\quad z = 3 \end{cases}$$

$\boxed{(2, -3, 3)}$

$y + 3(3) = 6$
$y + 9 = 6$
$-9 \quad -9$
$y = -3$

$2x - (-3) + 5(3) = 22$
$2x + 3 + 15 = 22$
$2x + 18 = 22$
$-18 \quad -18$
$\frac{2x}{2} = \frac{4}{2}$ $\quad x = 2$

Gaussian Elimination

To solve a system that is not in row-echelon form, first convert it to an equivalent system that is in row-echelon form by using the following operations.

Each of the following **row operations** on a system of linear equations produces an *equivalent* system of linear equations.
1. Interchange two equations.
2. Multiply one of the equations by a nonzero constant.
3. Add a multiple of one of the equations to another equation to replace the latter equation.

Examples:
2. Use row operations to solve the system of equations.

$$\begin{cases} 2x + y = 3 \\ x + 2y = 3 \end{cases}$$

$(-2)(R_2) \to R_2$ $\quad \begin{cases} 2x + y = 3 \\ -2x - 4y = -6 \end{cases}$

$\boxed{(1, 1)}$

$R_1 + R_2 \to R_2$ $\quad \begin{cases} 2x + y = 3 \\ -3y = -3 \end{cases}$

$-\frac{1}{3}R_2 \to R_2$ $\quad \begin{cases} 2x + y = 3 \\ y = 1 \end{cases}$

$2x + (1) = 3$
$-1 \quad -1$
$\frac{2x}{2} = \frac{2}{2}$
$x = 1$

Using Gaussian Elimination to Solve a System

3. $\begin{cases} x+y+z=6 \\ 2x-y-z=3 \\ 3x+y-z=2 \end{cases}$ $R_1 \leftrightarrow R_3$ $\begin{cases} 3x+y-z=2 \\ 2x-y-z=3 \\ x+y+z=6 \end{cases}$

$-2\cdot R_3$ $\begin{cases} 3x+y-z=2 \\ 2x-y-z=3 \\ -2x-2y-2z=-12 \end{cases}$

$R_2+R_3 \to R_3$ $\begin{cases} 3x+y-z=2 \\ 2x-y-z=3 \\ -3y-3z=-9 \end{cases}$

$6x+2y-2z=4$
$-6x+3y+3z=-9$

$2R_1+-3R_2 \to R_2$
$\frac{1}{3}R_3 \to R_3$ $\begin{cases} 3x+y-z=2 \\ 5y+z=-5 \\ -y-z=-3 \end{cases}$

$5y+z=-5$
$-5y-5z=-15$

$R_2+5\cdot R_3 \to R_3$ $\begin{cases} 3x+y-z=2 \\ 5y+z=-5 \\ -4z=-20 \end{cases}$

$-\frac{1}{4}R_3 \to R_3$ $\begin{cases} 3x+y-z=2 \\ 5y+z=-5 \\ z=5 \end{cases}$

$5y+5=-5$
$5y=-10$
$y=-2$

$3x+(-2)-(5)=2$
$3x-7=2$
$3x=9$
$x=3$

$\boxed{(3,-2,5)}$

4. $\begin{cases} x+y-2z=3 \\ 3x-2y+4z=1 \\ 2x-3y+6z=8 \end{cases}$

$-2x-2y+4z=-6$
$-3x-3y+6z=-9$

$-2\cdot R_1+R_3 \to R_3$ $\begin{cases} x+y-2z=3 \\ 3x-2y+4z=1 \\ -5y+10z=2 \end{cases}$

$-3\cdot R_1+R_2 \to R_2$ $\begin{cases} x+y-2z=3 \\ -5y+10z=-8 \\ -5y+10z=2 \end{cases}$

$-1\cdot R_3$ $\begin{cases} x+y-2z=3 \\ -5y+10z=-8 \\ 5y-10z=-2 \end{cases}$

$R_2+R_3 \to R_3$ $\begin{cases} x+y-2z=3 \\ -5y+10z=-8 \\ 0=-10 \end{cases}$

Not true!

$\boxed{\text{No Solution}}$

5. $\begin{cases} x+2y-7z=-4 \\ 2x+3y+z=5 \\ 3x+7y-36z=-25 \end{cases}$

$-2\cdot R_1 \to -2x-4y+14z=8$
$-3\cdot R_1 \to -3x-6y+21z=12$

$-2\cdot R_1+R_2 \to R_2$
$-3\cdot R_1+R_3 \to R_3$ $\begin{cases} x+2y-7z=-4 \\ -y+15z=13 \\ y-15z=-13 \end{cases}$

$R_2+R_3 \to R_3$ $\begin{cases} x+2y-7z=-4 \\ -y+15z=13 \\ 0=0 \end{cases}$

True

Equation 3 Depends on Equations 1 and 2

$x+2y-7z=-4$
$-y+15z=13 \to y=15z-13$

$x+2(15z-13)-7z=-4$
$x+30z-26-7z=-4$
$x+23z=22 \to x=-23z+22$

$z \to$ any value "a" then $y=15a-13$ and $x=-23a+22$

Infinitely many solutions w/ this form: $(-23a+22, 15a-13, a)$

p. 496: 8, 11, 15, 17, 19, 25, 31