Rational Function: a quotient of polynomial functions

$$f(x) = \frac{N(x)}{D(x)}$$

Domain: $\frac{1}{x}$ $\{x \mid x \neq 0\}$ OR $(-\infty,0)\cup(0,\infty)$

Example: Find the domain of each rational function and behavior near excluded values.

$D(x) \neq 0$

1. $f(x) = \frac{3x}{x-1}$

Domain: $x-1 \neq 0$, $x \neq 1$, $(-\infty,1)\cup(1,\infty)$

x	$f(x)$
2	6
1.9	$6.\overline{3}$
1.5	9
0	0
.5	-3
.9	-27

Behavior:
$f(x) \to \infty$ as $x \to 1^+$ ← towards 1 from the right
$f(x) \to -\infty$ as $x \to 1^-$ ← towards 1 from the left

2. $f(x) = \frac{3x}{x+10}$

$x \neq -10$
$(-\infty,-10)\cup(-10,\infty)$

x	$f(x)$
-9	-27
-9.1	$-30.\overline{3}$
-9.5	-57
-11	33
-10.9	$36.\overline{3}$
-10.5	63

$f(x) \to -\infty$ as $x \to -10^+$
$f(x) \to \infty$ as $x \to -10^-$

3. $f(x) = \frac{x+1}{(x+2)(x-6)}$

$x \neq -2, 6$
$(-\infty,-2)\cup(-2,6)\cup(6,\infty)$

* graph on calculator to see behavior

$f(x) \to \infty$ as $x \to -2^+$
$f(x) \to -\infty$ as $x \to -2^-$
$f(x) \to \infty$ as $x \to 6^+$
$f(x) \to -\infty$ as $x \to 6^-$

Vertical Asymptote:

$x = a$, the restriction from the domain (use $D(x)$)

$f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a^+$ or $x \to a^-$

The graph will NEVER cross a vertical asymptote.

Horizontal Asymptote: (End Behavior)

$y = b$

$f(x) \to b$ as $x \to \infty$ or as $x \to -\infty$

3 cases

① highest power in denominator

$\frac{x}{x^3}$

HA: $y = 0$

② highest power same in $N(x)$ and $D(x)$

$\frac{3x^2+2x-1}{5x^2}$

$y = \frac{3}{5}$ ratio of leading coefficients

③ highest power in Numerator $N(x)$

NO Horizontal Asymptote (slant or oblique)

Example: Find all vertical and horizontal asymptotes of the graph for each function.

4. $f(x) = \frac{3x^2+7x-6}{x^2+4x+3}$

V.A. $x^2+4x+3=0$
$(x+1)(x+3)=0$
$\boxed{x=-1,\ x=-3}$

H.A. $\frac{3x^2}{x^2}$
$\boxed{y=3}$
(case 2)

5. $f(x) = \frac{x}{x^2-9}$

V.A. $x^2-9=0$
$(x+3)(x-3)=0$
$\boxed{x=-3,\ x=3}$

H.A. $\frac{x}{x^2}$
$\boxed{y=0}$
(case 1)

Sketching the Graph of a Rational Function

$f(x) = \frac{N(x)}{D(x)}$, $D(x) \neq 0$

1. **Simplify *f* if possible.**
2. **Find and plot the *y*-intercept (if any) by evaluating *f(0)*.** let $x=0$
3. **Find the zeros of the numerator (if any) by solving the equation *N(x)=0*. Then plot the corresponding *x*-intercepts.** A fraction $=0$ when the numerator $=0$
4. **Find the zeros of the denominator (if any) by solving the equation *D(x)=0*. Then sketch the corresponding vertical asymptotes.** Domain, Vertical Asymptotes
5. **Find and sketch the horizontal asymptotes (if any) by using the rule for finding the horizontal asymptote of a rational function.**
6. **Plot at least one point *between* and one point *beyond* each *x*-intercept and Vertical asymptote.**
7. **Use smooth curves to complete the graph between and beyond the vertical asymptotes.**

Examples: Sketch the graph and state the domain for each function.

6. $f(x) = \frac{1}{x+3}$

y-int: $(0, \frac{1}{3})$ $\frac{1}{0+3}$

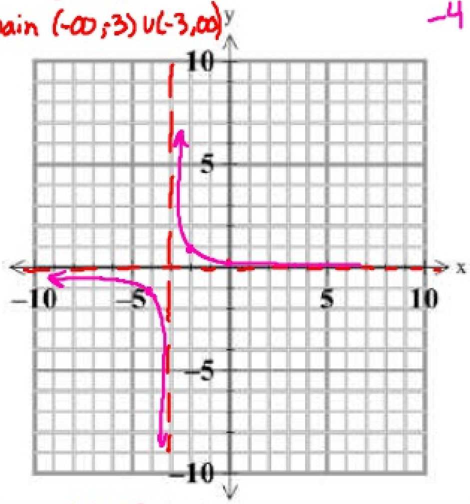
x-int: $N(x)=0$ $(\cancel{\ \ }, \cancel{0})$ $1 = \cancel{0}$? No x-int.

V.A $D(x)=0$
$x+3=0$
$x=-3$

H.A. $y=0$

x	y
-2	$\frac{1}{-2+3}=1$
-4	$\frac{1}{-4+3}=-1$

Domain $(-\infty,-3)\cup(-3,\infty)$



Asymptote Behavior

V.A. $f(x) \to -\infty$ as $x \to -3^-$
$f(x) \to \infty$ as $x \to -3^+$

HA. $f(x) \to 0$ as $x \to -\infty$
$f(x) \to 0$ as $x \to \infty$

7. $C(x) = \frac{3+2x}{1+x}$

y-int: $(0,3)$ $\frac{3+2(0)}{1+(0)}$

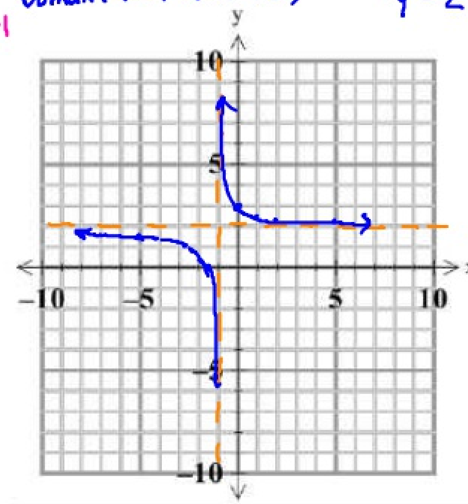
x-int: $(-\frac{3}{2}, 0)$
$3+2x=0$
$2x=-3$
$x=-\frac{3}{2}$

V.A. $1+x=0$
$x=-1$

H.A. $\frac{2x}{x}$
$y=2$

Domain $(-\infty,-1)\cup(-1,\infty)$

x	y
-5	$\frac{3+2(-5)}{1+(-5)}=\frac{-7}{-4}$ $1\frac{3}{4}$
5	$\frac{3+2(5)}{1+(5)}=\frac{13}{6}$
1	$2\frac{1}{2}$
3	$2\frac{1}{4}$



V.A. $C(x) \to -\infty$ as $x \to -1^-$
$C(x) \to \infty$ as $x \to -1^+$

H.A. $C(x) \to 2$ as $x \to -\infty$
$C(x) \to 2$ as $x \to \infty$

8. $f(x)=\dfrac{3x}{x^2+x-2}$

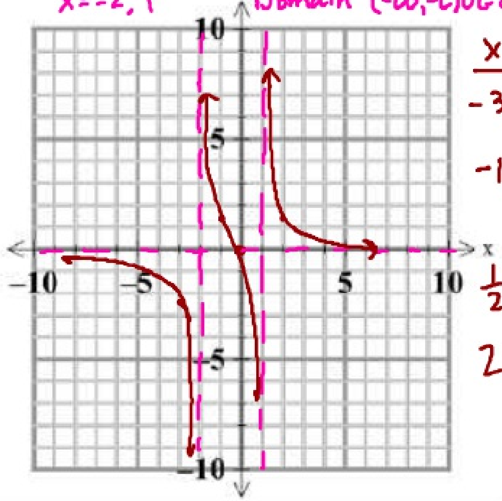
y-int: $(0,0)$ $\quad \dfrac{3(0)}{(0)^2+(0)-2}=0$

x-int: $(0,0)$ $\quad 3x=0$, $x=0$

VA: $(x^2+x-2)=0$
$(x+2)(x-1)=0$
$x=-2, 1$

H.A: $\dfrac{3x}{x^2}$ $\quad Y=0$

Domain $(-\infty,-2)\cup(-2,1)\cup(1,\infty)$



x	y
-3	$-\frac{9}{4}$
-1	$\frac{3}{2}$
$\frac{1}{2}$	-1.2
2	$\frac{3}{2}$

9. $f(x)=\dfrac{x^2-4}{x^2-x-6}=\dfrac{(x-2)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}}=\dfrac{x-2}{x-3}$

hole at $x=-2$

* get a hole when a factor cancels

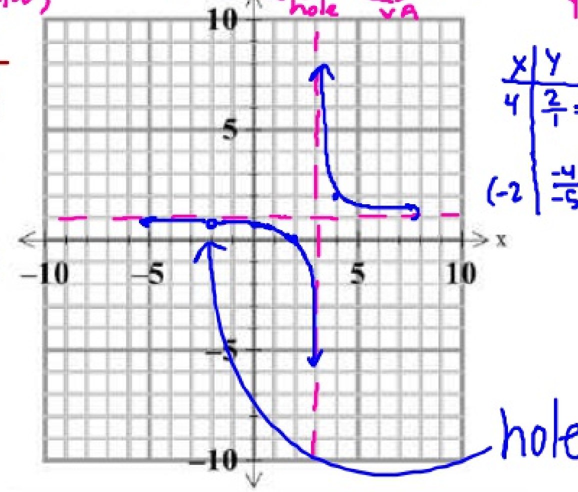
Y-int: $(0,\frac{2}{3})$ $\quad \dfrac{0-2}{0-3}$

x-int: $(2,0)$ $\quad x-2=0$, $x=2$

VA: $x-3=0$, $x=3$

HA: $\dfrac{x^2}{x^2}$ $\quad Y=1$

Domain $\underbrace{(-\infty,-2)\cup(-2,3)}_{\text{hole}}\underbrace{\cup(3,\infty)}_{\text{VA}}$



x	y
4	$\frac{2}{1}=2$
(-2	$\frac{-4}{-5}=\frac{4}{5}$ hole)

10. $f(x)=\dfrac{3x+4}{x^2+x-6}$

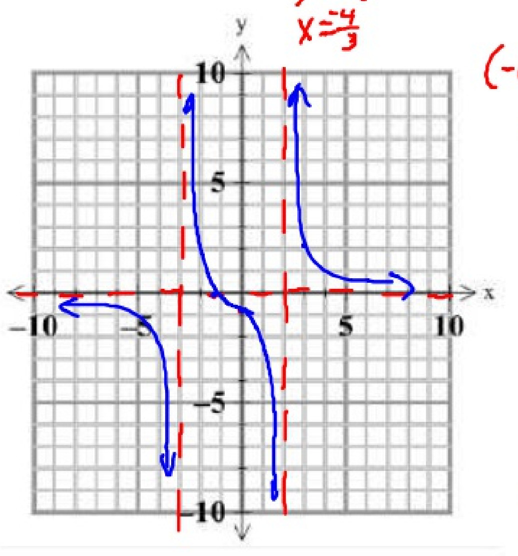
y-int $(0,-\frac{2}{3})$ $\quad \dfrac{3(0)+4}{0^2+0-6}=-\dfrac{4}{6}$

x-int $(-\frac{4}{3},0)$
$3x+4=0$
$-4\quad -4$
$\dfrac{3x}{3}=\dfrac{-4}{3}$
$x=\frac{-4}{3}$

Domain
$x^2+x-6\neq 0$
$(x+3)(x-2)\neq 0$
$x\neq -3, 2$
$(-\infty,-3)\cup(-3,2)\cup(2,\infty)$

V.A. $x=-3$, $x=2$

HA. $y=0$



x	y
3	$\frac{9+4}{9+3-6}=\frac{13}{6}$
-2	+
-5	−

Slant (Oblique) Asymptote $\frac{N(x)}{D(x)}$

Power in N(x) > Power in D(x)

$f(x) = \frac{x^2 - x}{x+1}$

$$
\begin{array}{r}
x - 2 + \frac{2}{x+1} \text{ insignificant} \\
x+1 \,\overline{\big)\, x^2 - x + 0} \\
-(x^2 + x) \\
\hline
-2x + 0 \\
-(-2x - 2) \\
\hline
2
\end{array}
$$

Slant Asymptote: $\boxed{y = x - 2}$

Examples Sketch the graph and state the domain for each function.

11. $f(x) = \frac{3x^2 + 1}{x}$

y-int: ~~(0,~~ undefined

$\cancel{\frac{3(0)^2+1}{0}}$ no y-int

x-int: $3x^2 + 1 = 0$

$\frac{3x^2}{3} = \frac{-1}{3}$

$\sqrt{x^2} = \sqrt{-\frac{1}{3}}$

no x-int.

Domain: $x \neq 0$

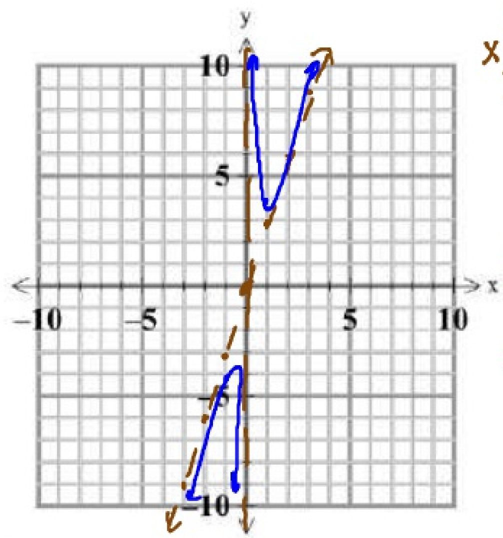
$(-\infty, 0) \cup (0, \infty)$

V.A. $x = 0$

Slant A. $y = 3x$

$$
\begin{array}{r}
3x + 0 + \frac{1}{x} \\
x \,\overline{\big)\, 3x^2 + 0x + 1} \\
-3x^2 \\
\hline
0x \\
-0x
\end{array}
$$

x	y
1	$\frac{4}{1} = 4$
-1	$\frac{4}{-1} = -4$



12. $f(x) = \frac{4x^2 - 2x + 1}{x - 1}$

y-int: $(0, -1)$

x-int: (,) None

Domain: $x - 1 \neq 0$

$x \neq 1$

$(-\infty, 1) \cup (1, \infty)$

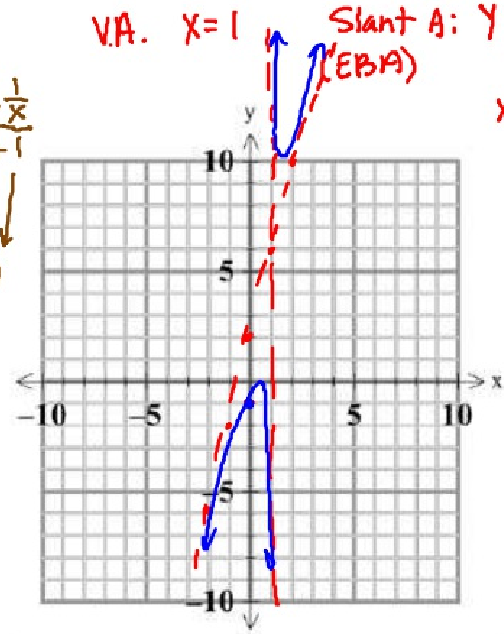
$4x^2 - 2x + 1 = 0$

$x = \frac{2 \pm \sqrt{4 - 16}}{8}$

V.A. $x = 1$

Slant A: $y = 4x + 2$ (EBA)

$$
\begin{array}{r}
4x + 2 \\
x-1 \,\overline{\big)\, 4x^2 - 2x + 1} \\
-(4x^2 - 4x) \\
\hline
2x + 1 \\
2x - 2
\end{array}
$$



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