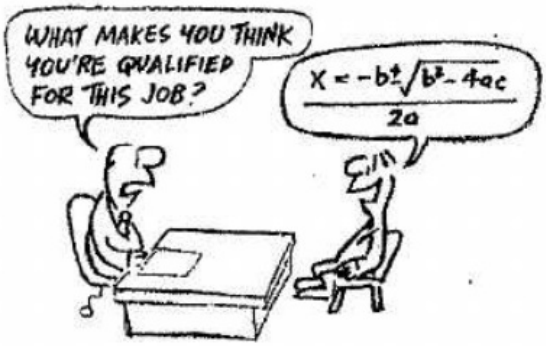
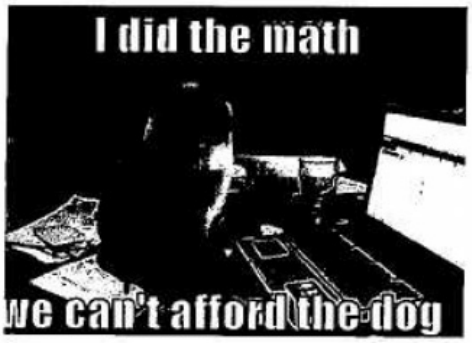
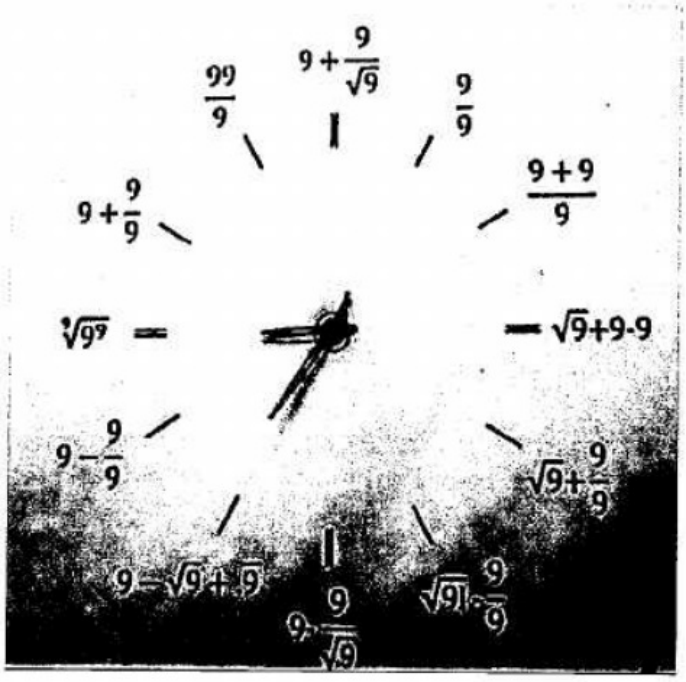
Math 1050

Test 2 Material

All About Quadratic and Rational Functions







Math 1050 -- 4.3 Notes

Quadratic Functions and Their Properties

Quadratic Function: A function that can be written in the form $f(x)=ax^2+bx+c$, where a, b, and c are real numbers and where $a\neq 0$. The domain of a quadratic function is the set of all real numbers.

Graphing a Quadratic Function Using Transformations

1. Begin with the parent function $f(x)=x^2$.
2. Rewrite the function in vertex form $f(x)=a(x-h)^2+k$ by completing the square.
3. Transform with the following:
   - a: If a is positive, the graph opens up. The y-coordinate of the vertex is a minimum value.
     If a is negative, the graph opens down. The y-coordinate of the vertex is a maximum value.
     If $|a|>1$, the graph is narrower than the graph of $f(x)=x^2$.
     If $|a|<1$, the graph is wider than the graph of $f(x)=x^2$.
   - h: h controls the horizontal shift (left and right).
   - k: k controls the vertical shift (up and down).

Vertex: (h,k) Axis of Symmetry: $x=h$

Completing the Square: Figuring out what constant to add to a binomial of the form x^2+bx to make it into a perfect square trinomial, then writing the result in factored form.

$a^2+2ab+b^2$

$(a+b)^2$

Completing the Square for the Binomial x^2+bx

1. Divide the coefficient of the x-term by 2. $\left(\text{Find } \frac{b}{2}\right)$.
2. Square the answer from step 1. $\left(\text{Find } \left(\frac{b}{2}\right)^2\right)$.
3. Add the result of step 2 to the binomial.
4. Rewrite as a perfect square: $\left(x+\frac{b}{2}\right)^2$.

Example: Add the proper constant to the binomial to make it into a perfect square trinomial. Then factor the trinomial.

$x^2+12x+\_\_\_$	$x^2-7x+\_\_\_$
$\frac{12}{2}=6$	$\frac{-7}{2}$
$6^2=36$	$\left(-\frac{7}{2}\right)^2=\frac{49}{4}$
$x^2+12x+36$	$x^2-7x+\frac{49}{4}$
$(x+6)^2$	$(x-\frac{7}{2})^2$

Examples: Add the proper constant to each binomial to make it into a perfect square trinomial. Then factor the trinomial.

a) $x^2+16x+\underline{64}=(x+8)^2$

d) $x^2-3x+\underline{\frac{9}{4}}=(x-\frac{3}{2})^2$

e) $x^2+\frac{4}{3}x+\underline{\frac{4}{9}}=(x+\frac{2}{3})^2$

$\frac{4}{3}\div 2$

$\frac{4}{3}\cdot\frac{1}{2}=\frac{4}{6}=\frac{2}{3}$

$\left(\frac{2}{3}\right)^2=\frac{4}{9}$

Writing $f(x)=ax^2+bx+c$ in Vertex Form

1. Group ax^2 and bx together in parentheses.
2. If $a \neq 1$, factor out a from ax^2+bx. Include a negative if the quadratic term is negative.
3. Complete the square (divide b by 2 and square the result). Add the answer inside the parentheses. Keep the equation balanced by adding or subtracting outside the parentheses. (You are adding 0 to one side of the equation.)
4. Write the expression inside the parentheses as a perfect square.

Examples: Write each equation in vertex form. Then find the vertex.

a) $f(x)=x^2-8x-5$ $\frac{8}{2}=4$ $4^2=16$ b) $f(x)=3x^2+6x+1$ $\frac{6}{2}=3$ $3^2=9$ c) $y=-x^2+4x-3$ $\frac{4}{2}=2$ $2^2=4$

$f(x)=(x^2-8x+16)-5-16$ add zero

$f(x)=(x-4)^2-21$

vertex: $(4,-21)$

$f(x)=3(x^2+2x)+1$

$f(x)=3(x^2+2x+1)+1-3$ add zero

$+3$

$f(x)=3(x+1)^2-2$

vertex: $(-1,-2)$

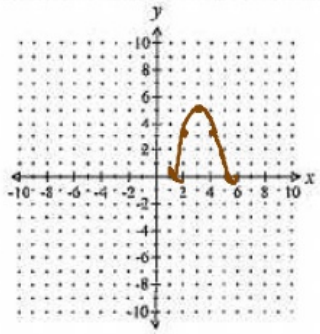
$y=-(x^2-4x+4)-3+4$ add zero

-4

$f(x)=-(x-2)^2+1$

vertex: $(2,1)$

Example: Rewrite the function $f(x)=-2x^2+12x-13$ in vertex form by completing the square. Find the vertex and axis of symmetry, then draw the graph. Find the maximum or minimum value.



$f(x)=-2(x^2-6x+9)-13+18$ add zero

-18

$f(x)=-2(x-3)^2+5$

vertex: $(3,5)$

axis of symmetry: $x=3$

maximum value $=5$

The Vertex Formula

By completing the square, we can rewrite $f(x)=ax^2+bx+c$ as $f(x)=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$.

This gives us a quick way to find the vertex when the equation is in standard form:

- The x-coordinate of the vertex is $-\frac{b}{2a}$.
- To find the y-coordinate, plug the x-coordinate into the original equation.

Properties of the Graph of a Quadratic Function

For the function, $f(x)=ax^2+bx+c$, $a \neq 0$:

Vertex: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ **Axis of Symmetry:** The line $x=-\frac{b}{2a}$

Parabola opens up if $a > 0$; the vertex is a minimum point.
Parabola opens down if $a < 0$; the vertex is a maximum point.

$x = \frac{-b}{2a}$

Examples: Use the vertex formula to locate the vertex and axis of symmetry of the parabola. Determine whether the quadratic function has a maximum or minimum value, then find the value.

a) $f(x) = x^2 - 6x - 8$

$x = \frac{-(-6)}{2(1)} = 3$

$f(3) = (3)^2 - 6(3) - 8$

$= 9 - 18 - 8 = -17$

vertex: $(3, -17)$
axis of symm: $x = 3$
minimum $= -17$

b) $f(x) = -4x^2 + 2x + 1$

$x = \frac{-(2)}{2(-4)} = \frac{1}{4}$

$f\left(\frac{1}{4}\right) = -4\left(\frac{1}{4}\right)^2 + 2\left(\frac{1}{4}\right) + 1$

$= -4\left(\frac{1}{16}\right) + 2\left(\frac{1}{4}\right) + 1$

$= -\frac{1}{4} + \frac{1}{2} + 1 = \frac{5}{4}$

vertex: $\left(\frac{1}{4}, \frac{5}{4}\right)$
axis of symm.: $x = \frac{1}{4}$
maximum $= \frac{5}{4}$

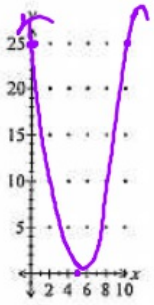
Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

The y-intercept is the value of f at $x = 0$; that is, the y-intercept is $f(0) = c$.

The x-intercepts, if any, are found by solving the quadratic equation $ax^2 + bx + c = 0$.

Discriminant	Graph of $f(x) = ax^2 + bx + c$	Solutions of $ax^2 + bx + c = 0$
$b^2 - 4ac > 0$	Two x-intercepts	Two real solutions
$b^2 - 4ac = 0$	One x-intercept	One real solution
$b^2 - 4ac < 0$	No x-intercepts	No real solutions

Example: Graph the function $f(x) = x^2 - 10x + 25$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any. Determine the domain and range of f and determine where f is increasing and where it is decreasing.



opens up

$f(x) = (x-5)^2$

vertex: $(5, 0)$

axis of symmetry: $x = 5$

x-int: $(5, 0)$

$0 = (x-5)^2$

$0 = x - 5$ so $x = 5$

y-int: $(0, 25)$

$f(x) = (0-5)^2$

$= 25$

Domain: $(-\infty, \infty)$ or $\mathbb{R}$

Range: $[0, \infty)$ or $\{y \mid y \geq 0\}$

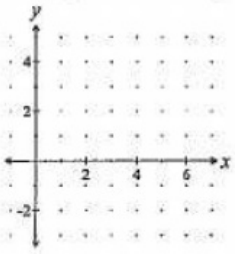
Inc: $(5, \infty)$ or $\{x \mid x > 5\}$

Dec: $(-\infty, 5)$ or $\{x \mid x < 5\}$

Writing a Quadratic Equation when You Know the Vertex and Another Point

1. Use vertex form: $y = a(x-h)^2 + k$
2. Plug in the vertex for h and k.
3. Plug in the other point for x and y (or $f(x)$).
4. Simplify and solve for a. (Don't forget order of operations.)
5. Write your final answer by plugging a, h, and k back into vertex form.

Example: Find the quadratic function whose vertex is $(3, -2)$ and whose y-intercept is 4. Graph the function.



$y = a(x-3)^2 - 2$ (h, k)

find this

(0, 4) (x, y)

$4 = a(0-3)^2 - 2$

$4 = a(-3)^2 - 2$

$4 = 9a - 2$

$6 = 9a$

$\frac{6}{9} = a$

$a = 2/3$

$y = \frac{2}{3}(x-3)^2 - 2$

	Standard Form	Vertex Form	Factored Form
Equation	$y=ax^2+bx+c$	$y=a(x-h)^2+k$	$y=a(x-p)(x-q)$
Vertex	Complete the square and write in vertex form. -or- $x=\frac{-b}{2a}$ Plug the x-coordinate into the equation to get the y-coordinate.	(h,k)	Find average of p and q. $x=\frac{p+q}{2}$ (The x-coordinate of the vertex is at the midpoint of the x-intercepts.) Plug the x-coordinate into the equation to get the y-coordinate.
***y*-intercept**	c (Replace x with zero. Solve for y.)	Replace x with zero. Solve for y.	Replace x with zero. Solve for y.
***x*-intercepts (roots, zeros, solutions)**	Replace y with zero. Solve for x by factoring or quadratic formula.	Replace y with zero. Solve for x by isolating the perfect square and using the square root principle. (Don't forget the $\pm$.)	p and q (Replace y with zero. Solve for x using the zero product property.)

For all forms:

Direction of Opening	Up if a is positive Down if a is negative
Vertical Stretch	a
Counting Pattern (Shortcut)	Start at the vertex. Find more points by counting: $\leftrightarrow 1, \uparrow a$ $\leftrightarrow 1, \uparrow 3a$ $\leftrightarrow 1, \uparrow 5a$ $\leftrightarrow 1, \uparrow 7a$, etc. (If a is negative, move down instead of up.)

Math 1050 – 4.4 Notes

Quadratic Models; Building Quadratic Functions from Data

Example: Maximizing Revenue

In economics, revenue R, in dollars, is defined by the amount of money received from the sale of an item and is equal to the selling price p, in dollars, of the item times the number x of units actually sold. That is, $R = xp$.

The price p in dollars and the quantity x sold of a certain product obey the demand equation

$$p = -\frac{1}{3}x + 100, \quad 0 \le x \le 300.$$

a) Express the revenue R as a function of x.
b) What is the revenue if 100 units are sold?
c) What quantity x maximizes revenue? What is the maximum revenue?
d) What price should the company charge to maximize revenue?

a) $R = xp$

$R = x\left(-\frac{1}{3}x + 100\right)$

$\boxed{R(x) = -\frac{1}{3}x^2 + 100x}$

b) $R(100) = -\frac{1}{3}(100)^2 + 100(100)$

$= \boxed{\$16666.67}$

c) $x = \frac{-b}{2a}$

$x = \frac{-100}{2\left(-\frac{1}{3}\right)}$

$= \frac{-100}{-\frac{2}{3}}$

$= -100 \cdot \frac{-3}{2}$

$= \boxed{150 \text{ units}}$

$R(150) = -\frac{1}{3}(150)^2 + (100)(150)$

$= \boxed{\$7500}$

The maximum revenue is \$7500 when 150 units are sold.

d) $R = xp$

$7500 = 150p$

$p = \boxed{\$50}$

or

$p = -\frac{1}{3}(150) + 100$

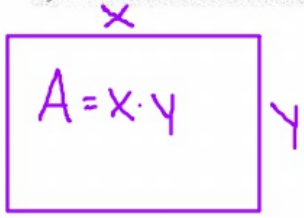
$= -50 + 100$

$= \$50$

The maximum revenue will be reached if the price is \$50.

Example: Maximizing the Area Enclosed by a Fence

Beth has 3000 feet of fencing available to enclose a rectangular field.
a) Express the area A of the rectangle as a function of x, where x is the length of the rectangle.
b) For what value of x is the area largest?
c) What is the maximum area?
d) What are the dimensions of the field with the maximum area?



$P = 3000$

$P = 2x + 2y$

$2x + 2y = 3000$

$2y = 3000 - 2x$

$y = 1500 - x$

a) $A = x \cdot y$

$A = x(1500 - x)$

$A = 1500x - x^2$

$A(x) = -x^2 + 1500x$

b) $x = \frac{-b}{2a}$

$x = \frac{-1500}{2(-1)}$

$= \boxed{750 \text{ feet}}$

Area is maximized when $x = 750$ feet.

c) $A(750) = -(750)^2 + 1500(750)$

$= \boxed{562{,}500 \text{ ft}^2}$

The maximum area is $562{,}500 \text{ ft}^2$

d) $y = 1500 - 750$

$y = 750$

The field with the maximum area is

$\boxed{750\text{ft} \times 750\text{ft}}$

Example: Analyzing the Motion of a Projectile

A projectile is fired at an inclination of 45° to the horizontal, with a muzzle velocity of 100 feet per second. The height h of the projectile is given by $h(x) = \frac{-32x^2}{(100)^2} + x$, where x is the horizontal distance of the projectile from the firing point.

a) At what horizontal distance from the firing point is the height of the projectile a maximum?
b) Find the maximum height of the projectile.
c) At what horizontal distance from the firing point will the projectile strike the ground?
d) Using a graphing calculator, graph the function h, $0 \le x \le 350$.
e) Using a graphing calculator, verify the results obtained in parts b) and c).

$h(x) = -.0032x^2 + x$

a) $x = \frac{-b}{2a}$

$x = \frac{-1}{2(-.0032)} = \boxed{156.25 \text{ ft}}$

b) $h(156.25) = -.0032(152.25)^2 + 156.25$

$= \boxed{78.125 \text{ ft}}$

The projectile reached a maximum of 78.125 ft when it had traveled 156.25 horizontally.

c) when the projectile hits the ground, $h = 0$

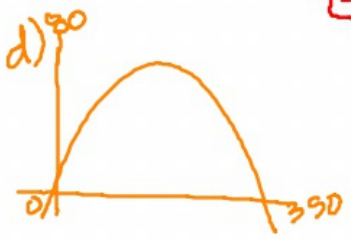
$0 = -.0032x^2 + x$

$0 = x(-.0032x + 1)$

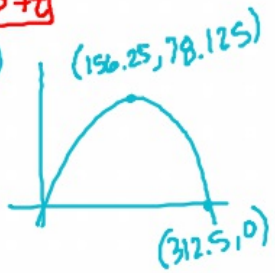
$x = 0$ or $x = \frac{1}{.0032}$

$= \boxed{312.5 \text{ ft}}$

d)

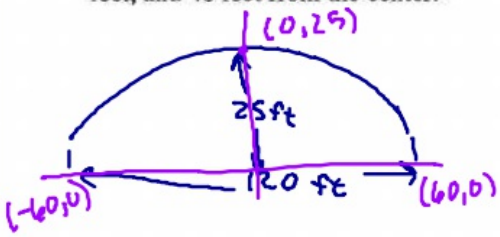


e)



Example: A Parabolic Arch

A parabolic arch has a span of 120 feet and a maximum height of 25 feet. Choose a suitable rectangular coordinate axes and find the equation of the parabola. Then calculate the height of the arch at points 10 feet, 20 feet, and 40 feet from the center.



$f(x) = a(x-h)^2 + k$ $\quad (h, k)$

$f(x) = a(x-0)^2 + 25$ $\quad (0, 25)$

$0 = a(60)^2 + 25$ $\quad (x, y)$

$(60, 0)$

$\frac{-25}{3600} = \frac{3600a}{3600}$

$\frac{-1}{144} = a$

$-.0069$

$\boxed{f(x) = -\frac{1}{144}x^2 + 25}$

$f(10) = -.0069(10)^2 + 25$

$= \boxed{24.31 \text{ ft}}$

$f(20) = -.0069(20)^2 + 25$

$= \boxed{22.24 \text{ ft}}$

$f(40) = -.0069(40)^2 + 25$

$= \boxed{13.96 \text{ ft}}$

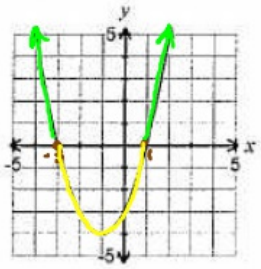
Math 1050 – 4.5 Notes

Inequalities Involving Quadratic Functions

Examples: Solve each inequality using the graph of $f(x)=x^2+2x-3$.

Notice that each of these inequalities involves the value of x^2+2x-3, which is represented by the y-coordinate of the graph. In each case, we are trying to figure out what x-values (x-coordinates) make the inequality true. When trying to find where $x^2+2x-3>0$, we are trying to figure out what x-coordinates have a y-coordinate that is bigger than zero—in other words, *where is the graph above the x-axis?*

$f(x)=x^2+2x-3$



a) $x^2+2x-3>0$ $y>0$ (above x-axis)

$(-\infty,-3)\cup(1,\infty)$

or $\{x \mid x<-3 \text{ or } x>1\}$

b) $x^2+2x-3\geq 0$ $y\geq 0$ (on or above x-axis)

$(-\infty,-3]\cup[1,\infty)$

or $\{x \mid x\leq -3 \text{ or } x\geq 1\}$

c) $x^2+2x-3<0$ $y<0$ (below x-axis)

$(-3,1)$

or $\{x \mid -3<x<1\}$

d) $x^2+2x-3\leq 0$ $y\leq 0$ (on or below x-axis)

$[-3,1]$

or $\{x \mid -3\leq x\leq 1\}$

Solving a Quadratic Inequality in One Variable Graphically:

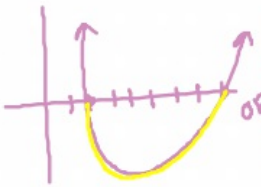
1. Write the inequality in standard form. Replace the inequality sign with an equal sign and solve the equation $ax^2+bx+c=0$ by factoring, completing the square, or using the quadratic formula. This gives you the x-intercepts of the graph of $y=ax^2+bx+c$.
2. Graph the related function $y=ax^2+bx+c$. The graph does not have to be very detailed. A rough sketch of a parabola opening in the correct direction with the correct x-intercepts is all you need.
3. The solutions of $ax^2+bx+c>0$ are the x-values for which the graph is **above** the x-axis.
   The solutions of $ax^2+bx+c\geq 0$ are the x-values for which the graph is **on or above** the x-axis.
   The solutions of $ax^2+bx+c<0$ are the x-values for which the graph is **below** the x-axis.
   The solutions of $ax^2+bx+c\leq 0$ are the x-values for which the graph is **on or below** the x-axis.
4. If the inequality involves $\leq$ or $\geq$, the x-intercepts **are included** in the solution set (use brackets).
   If the inequality involves $<$ or $>$, the x-intercepts **are not included** in the solution set (use parentheses).

Example: Solve each inequality and graph the solution set on a number line.

a) $x^2-10x+16\leq 0$ on or below x-axis

$(x-8)(x-2)\leq 0$

x-int: 8, 2



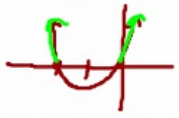
$[2,8]$

or $\{x \mid 2\leq x\leq 8\}$

b) $x^2+2x>0$ above x-axis

$x(x+2)>0$

x-int: 0, -2



$(-\infty,-2)\cup(0,\infty)$

or $\{x \mid x<-2 \text{ or } x>0\}$

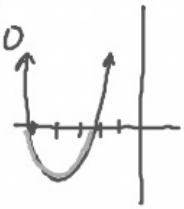
c) $-x^2-7x\geq 10$ on or above x-axis

$0\geq x^2+7x+10$

$x^2+7x+10\leq 0$

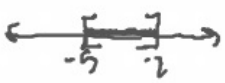
$(x+2)(x+5)\leq 0$

x-int: -2, -5



$[-5,-2]$

or $\{x \mid -5\leq x\leq -2\}$



d) $3x^2+10x-8<0$ below x-axis

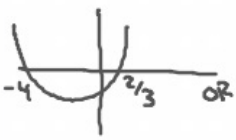
$\times$ $+$
-24 10

$3x^2+12x-2x-8<0$

$3x(x+4)-2(x+4)<0$

$(3x-2)(x+4)<0$

x-int: $\frac{2}{3}, -4$



$(-4, \frac{2}{3})$

OR $\{x \mid -4<x<\frac{2}{3}\}$

e) $x^2-4x+5\le 0$ on or below x-axis

$x=\frac{4\pm\sqrt{16-20}}{2}$

$x=\frac{4\pm\sqrt{-4}}{2}=\frac{4\pm 2i}{2}=2\pm i$

no x-int

Graph is never below x-axis

$\varnothing$

No Solution

f) $x^2+6x+9>0$ above x-axis

$(x+3)^2>0$

x-int: -3

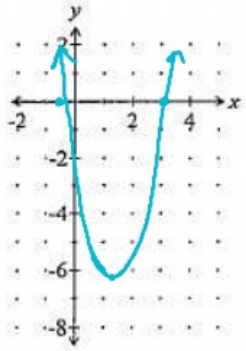
$(-\infty,-3)\cup(-3,\infty)$

OR $\{x \mid x\neq -3\}$

Example: Solve the inequality $2x^2<5x+3$ and graph the solution set.
Method 1: Rearrange the inequality so that 0 is on the right side.

$\times$ $+$
-6 -5

$-6, 1$



$2x^2-5x-3<0$

$2x^2-6x+x-3<0$

$2x(x-3)+1(x-3)<0$

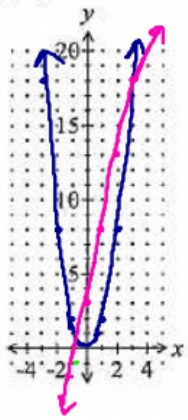
$(2x+1)(x-3)<0$

x-int: $-\frac{1}{2}, 3$

$(-\frac{1}{2}, 3)$

OR $\{x \mid -\frac{1}{2}<x<3\}$

Method 2: Find the intersections of the two functions and determine where $f(x)=2x^2$ is less than $f(x)=5x+3$.



$2x^2<5x+3$

$(-\frac{1}{2}, 3)$

OR $\{x \mid -\frac{1}{2}<x<3\}$

Math 1050 -- 5.1 Notes

Polynomial Functions and Models

A ***polynomial function*** is a function in the form $f(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x+a_0$ where a_n, a_{n-1}, ..., a_1, a_0 are real numbers (coefficients) and n is a nonnegative integer. The domain of a polynomial function is the set of all real numbers. The ***degree*** of a polynomial is the largest power of x that appears. The zero polynomial $f(x)=0$ is not assigned a degree.

$a_0 x^0$

Example: Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not.

a) $f(x)=5+2x^2-8x^3$

Yes, degree 3

b) $f(x)=4+3\sqrt{x}$ $3\sqrt{x} = 3x^{1/2}$

no, $\sqrt{}$ not allowed to have fraction exponents

c) $f(x)=-2x^3(x-1)^2$

$f(x) = -2x^3(x^2-2x+1)$
$= -2x^5+4x^4-2x^3$

yes, degree 5

d) $f(x)=0$

yes, no degree

e) $f(x)=\dfrac{x^2-1}{x+4}$

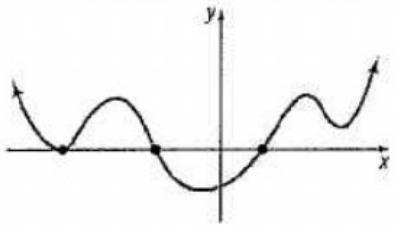
no, rational function

domain is not $\mathbb{R}$

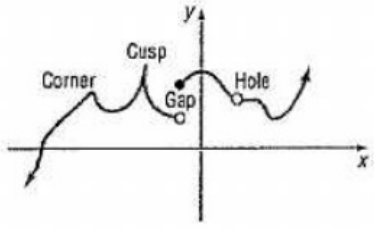
f) $f(x)=9$

yes, degree 0

A polynomial function is smooth and continuous. It does not contain corners, cusps, gaps, or holes.



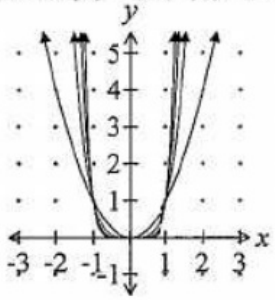
(a) Graph of a polynomial function: smooth, continuous



(b) Cannot be the graph of a polynomial function

A ***power function of degree n*** is a monomial of the form $f(x)=ax^n$, where a is a real number, $a\neq 0$, and n is an integer greater than 0.

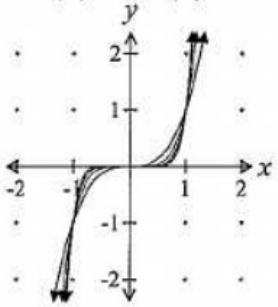
Example: Compare the even power functions $f(x)=x^2$, $f(x)=x^4$, $f(x)=x^6$, and $f(x)=x^8$.



Properties of Power Functions $f(x)=x^n$, n Is an Even Integer

1. f is an even function, so its graph is symmetric with respect to the y-axis.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contain the points (-1, 1), (0, 0), and (1, 1).
4. As the exponent n increases in magnitude, the graph becomes more vertical when $x<-1$ or $x>1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

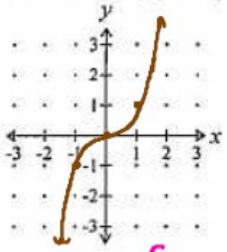
Example: Compare the odd power functions $f(x)=x^3$, $f(x)=x^5$, $f(x)=x^7$, and $f(x)=x^9$.

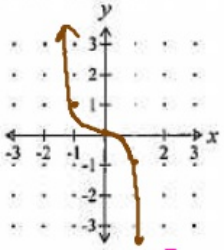


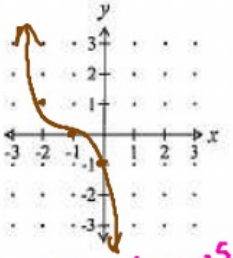
Properties of Power Functions $f(x)=x^n$, n Is an Odd Integer

1. f is an odd function, so its graph is symmetric with respect to the origin.
2. The domain and the range are the set of all real numbers.
3. The graph always contain the points (-1, -1), (0, 0), and (1, 1).
4. As the exponent n increases in magnitude, the graph becomes more vertical when $x<-1$ or $x>1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

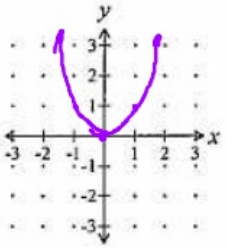
Example: Graph $f(x)=-(x+1)^5$ using transformations.

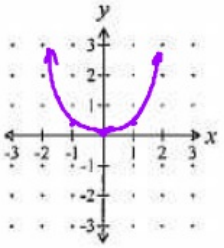


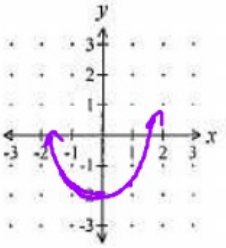




Example: Graph $f(x)=\frac{1}{3}x^4-2$ using transformations.







If f is a function and r is a real number for which $f(r)=0$, then r is called a ***real zero*** of f. The real zeros of a polynomial function are the x-intercepts of its graph, and they are found by solving the equation $f(x)=0$.

The following statements are equivalent:

1. r is a solution of $f(x)=0$.
2. r is a real zero of a polynomial function f.
3. r is an x-intercept of the graph of f.
4. $x-r$ is a factor of f.

Example:

a) Find a polynomial of degree 3 whose zeros are –2, 4, and 5.

b) Use a graphing calculator to graph the polynomial and verify the results of part a).

a) $f(x)=(x+2)(x-4)(x-5)$

$=(x+2)(x^2-9x+20)$

$=x^3-9x+20x+2x^2-18x+40$

$f(x)=x^3-7x^2+2x+40$

b)

If the same factor $x-r$ occurs more than once, r is called a ***repeated***, or ***multiple zero*** of f. More precisely, if $(x-r)^m$ is a factor of a polynomial f and $(x-r)^{m+1}$ is not a factor of f, then r is called a ***zero of multiplicity m*** of f.

Example: Identify the zeros and their multiplicities for the polynomial $f(x)=3(x-4)(x+1)^3(x-2)^2$.

zeros	multiplicity
4	1
-1	3
2	2

$y=3x^6$

Multiplicity Rules:

If r is a zero of even multiplicity:

1. The sign of $f(x)$ does not change from one side to the other side of r.
2. The graph of f **touches** (bounces) the x-axis at r.

$y=x^2$

If r is a zero of odd multiplicity:

1. the sign of $f(x)$ changes from one side to the other of r.
2. The graph of f **crosses** the x-axis at r.

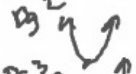
$y=x$

$y=x^3$

Behavior Near a Zero: This will not be tested. Skip it on your homework. If you are interested, see p. 327.

Turning Points: The points at which a graph changes direction (the local maxima and local minima) are called turning points. If f is a polynomial function of degree n, then f has at most $n-1$ turning points. If the graph of a polynomial f has $n-1$ turning points, the degree of f is at least n.

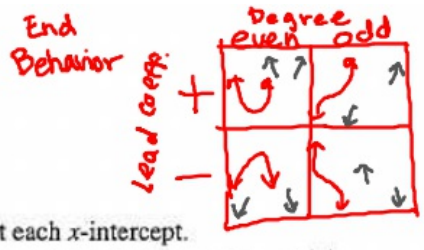
Deg 2



Deg 3

Deg 4

End Behavior: The behavior of the graph of a function for large values of x, either positive or negative, is referred to as its end behavior. For large values of x, either positive or negative, the graph of the polynomial function $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ resembles the graph of the power function $y=a_nx^n$.



Example: Graph the polynomial $f(x)=x^2(x+3)$.

a) Find the x-intercepts and y-intercept of the graph of f.

b) Determine whether the graph crosses or touches the x-axis at each x-intercept.

c) End behavior: Find the power function that the graph of f resembles for large values of $|x|$.

d) Determine the maximum number of turning points on the graph of f.

c) Locate other points on the graph and connect all the points plotted with a smooth continuous curve.

a) x-int: $0=x^2(x+3)$

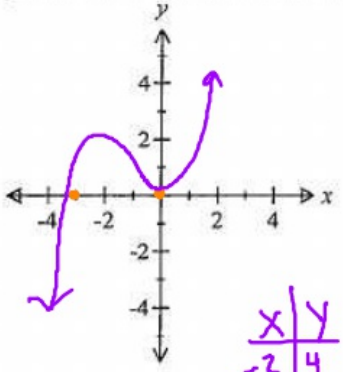
$x=0$ mult. 2 $\quad x=-3$ mult. 1

y-int: $f(0)=0^2(0+3)=0$

b) touches @ $x=0$
crosses @ $x=-3$

c) $y=x^3$ ↑ ↓

d) degree 3
max turning points is 2



x	y
-2	4
-1	2
1	4
2	20

Example: Graph the polynomial $f(x)=(x+1)^2(x-3)(x-1)$.

a) Find the x-intercepts and y-intercept of the graph of f.

b) Determine whether the graph crosses or touches the x-axis at each x-intercept.

c) End behavior: Find the power function that the graph of f resembles for large values of $|x|$.

d) Determine the maximum number of turning points on the graph of f.

c) Locate other points on the graph and connect all the points plotted with a smooth continuous curve.

a) x-int: $0=(x+1)^2(x-3)(x-1)$

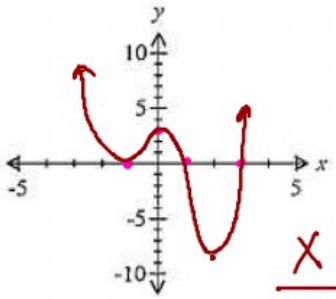
$x=-1, 3, 1$
mult 2 mult. 1

y-int: $f(0)=(0+1)^2(0-3)(0-1)=3$

b) touches @ $x=-1$
crosses @ $x=3, 1$

c) $y=x^4$ ↑ ↑

d) degree 4
max turning points is 3



x	y
-2	15
2	-9
4	75

Example: Graph the polynomial $f(x)=-\frac{1}{2}x(x^2+1)(x-2)^2$.

a) Find the x-intercepts and y-intercept of the graph of f.

b) Determine whether the graph crosses or touches the x-axis at each x-intercept.

c) End behavior: Find the power function that the graph of f resembles for large values of $|x|$.

d) Determine the maximum number of turning points on the graph of f.

c) Locate other points on the graph and connect all the points plotted with a smooth continuous curve.

a) x-int: $0=-\frac{1}{2}x(x^2+1)(x-2)^2$

no zeros

$x=0$ mult. 1 $x=2$ mult. 2

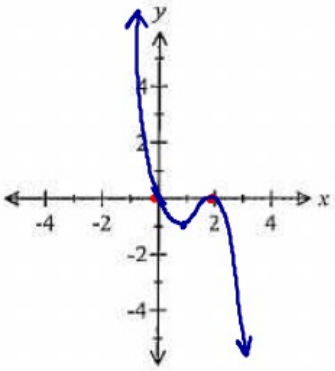
y-int: $f(0)=-\frac{1}{2}(0)(0^2+1)(1-2)^2=0$

b) Crosses @ $x=0$

touches @ $x=2$

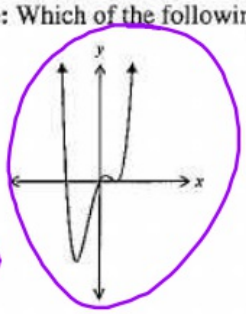
c) $y=-\frac{1}{2}x^5$ ↑ ↓

d) Degree 5 max turning points is 4



x	y
-1	9
1	-1
3	-15

Example: Which of the following could be the graph of $f(x)=x^4-3x^2+2x$?

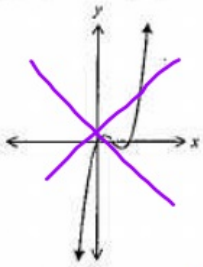


f(0)=0

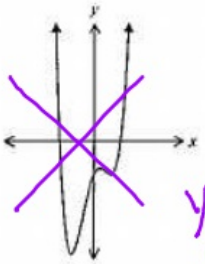
y-int = 0

Degree 4 ↑ ↑

Max turns 3



Wrong E.B



y-int is not 0

Math 1050 – 5.2 Notes

Properties of Rational Functions

A ***rational function*** is a function of the form $R(x)=\frac{p(x)}{q(x)}$, where p and q are polynomial functions and q is not the zero polynomial.

Finding the Domain of a Rational Function
The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Examples: Find the domain of each rational function.

a) $R(x)=\frac{2x^2-3}{x+3}$

$x+3\neq 0$

$\{x \mid x\neq -3\}$

b) $R(x)=\frac{x+3}{x^2-9}=\frac{x+3}{(x+3)(x-3)}$

$x^2-9\neq 0$

$(x+3)(x-3)\neq 0$

$\{x \mid x\neq -3, 3\}$

c) $R(x)=\frac{x^2}{x^2+7x+12}=\frac{x^2}{(x+3)(x+4)}$

$(x+3)(x+4)\neq 0$

$\{x \mid x\neq -4, -3\}$

★ **Note:** It is important to understand that $R(x)=\frac{x+3}{x^2-9}$ and $f(x)=\frac{1}{x-3}$ are not the same. They have different domains. Their graphs are nearly identical, but the graph of the first function has a hole in it at $x=-3$, while the graph of the second function does not.

A rational function $R(x)=\frac{p(x)}{q(x)}$ is in ***lowest terms*** if $p(x)$ and $q(x)$ have no common factors.

Finding the Intercepts of a Rational Function

★ **Warning:** Always write the rational function in lowest terms before finding the x-intercepts of the graph. Otherwise, you may end up listing values that are actually holes in the graph. When finding the y-intercepts, remember that if zero is not in the domain, there is no y-intercept

To find the x-intercepts (real zeros) of the graph of a rational function, we set $R(x)=0$ and solve for x.

Notice that if $R(x)=\frac{p(x)}{q(x)}=0$, $p(x)$ must equal zero. It is the numerator that tells us about the x-intercepts.

- To find the ***x*-intercepts** (real zeros), simplify the rational function and set the numerator equal to zero.
- To find the ***y*-intercept**, plug $x=0$ into either the simplified or the unsimplified function. If zero is not in the domain of the function, there is no y-intercept.

Examples: Find the x- and y- intercepts of each rational function.

a) $R(x)=\frac{x+2}{x^2-16}=\frac{x+2}{(x+4)(x-4)}$

$x+2=0$

$x=-2$

$(-2,0)$

$R(0)=\frac{0+2}{0^2-16}=-\frac{2}{16}$

$(0,-\frac{1}{8})$

b) $R(x)=\frac{x^2-2x-35}{x^2+11x+30}=\frac{(x-7)(x+5)}{(x+6)(x+5)}$

$x-7=0$

$x=7$

$(7,0)$

$R(0)=\frac{0-7}{0+6}=-\frac{7}{6}$

$(0,-\frac{7}{6})$

c) $R(x)=\frac{x}{x^2+8x}=\frac{x^1}{x(x+8)}$

$x\neq 0$

no x-int

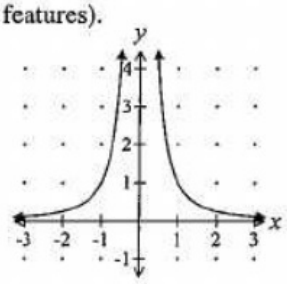
no y-int

zero not in domain

Domain $\{x \mid x\neq 0, -8\}$

$y = \frac{1}{x}$

Example: Analyze the graph of $H(x) = \frac{1}{x^2}$. (Find the domain, range, intercepts, and any other important features).



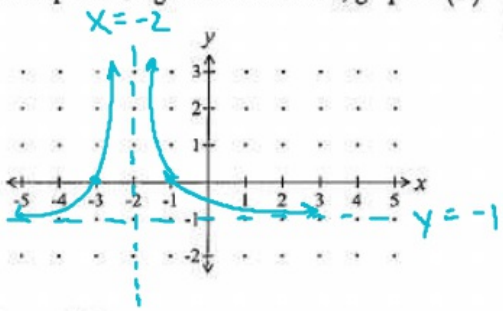
Domain: $\{x \mid x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$

Range: $\{y \mid y > 0\}$ or $(0, \infty)$

No intercepts

Symmetric over y-axis (even)

Example: Using transformations, graph $R(x) = \frac{1}{(x+2)^2} - 1$.



Left 2 Down 1

Asymptotes

In the previous example, notice that as the values of x become more negative, that is, as x becomes ***unbounded in the negative direction*** $(x \to -\infty$, read "as x approaches negative infinity") the values of $R(x)$ approach -1.

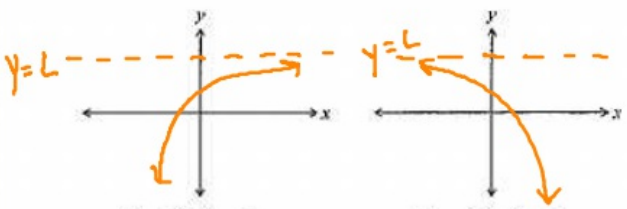
We conclude the following:

1. As $x \to -\infty$, the values of $R(x) \to -1$.
2. As $x \to -2$, the values of $R(x) \to \infty$.
3. As $x \to \infty$, the values of $R(x) \to -1$.

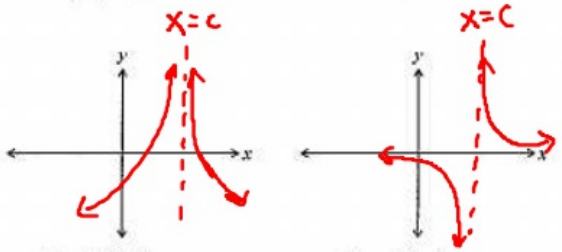
The graph goes near, but does not reach the lines $x = -2$ and $y = -1$. These lines are called ***asymptotes***.

Let R denote a function.

- If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a ***horizontal asymptote*** of the graph of R.
- If, as x approaches some number c, the values of $R(x)$ approach ∞ or $-\infty$, then the line $x = c$ is a ***vertical asymptote*** of the graph of R.



$\lim_{x \to \infty} R(x) = L$ $\lim_{x \to -\infty} R(x) = L$



$\lim_{x \to c^-} R(x) = \infty$ from the left $\lim_{x \to c^+} R(x) = \infty$ from the right

$\lim_{x \to c^-} R(x) = -\infty$ $\lim_{x \to c^+} R(x) = \infty$

There is a third type of asymptote called an ***oblique asymptote***. An oblique asymptote occurs when the graph's end behavior follows an oblique line (a linear function of the form $y = ax + b,\ a \neq 0$). A graph may also approach some function such as a parabola.

★ **Note:** The graph of a function may intersect a horizontal or oblique asymptote at some other point, but the graph will never intersect a vertical asymptote.

Find the Vertical Asymptotes of a Rational Function

A rational function $R(x) = \frac{p(x)}{q(x)}$, in *lowest terms* will have a vertical asymptote at $x = r$ if r is a real zero of the denominator q. That is, if $x - r$ is a factor of the denominator q of the rational function in lowest terms, it will have the vertical asymptote $x = r$.

- **Warning:** If the rational function is not in lowest terms, this theorem will result in an incorrect listing of vertical asymptotes.

Example: Find the vertical asymptotes, if any, of the graph of each rational function.

a) $R(x) = \frac{x+1}{x^2-9} = \frac{x+1}{(x+3)(x-3)}$

$(x+3)(x-3) = 0$

$x = -3, x = 3$

b) $R(x) = \frac{x+2}{x^2-3x-10} = \frac{x+2}{(x+2)(x-5)}$

$x - 5 = 0$

$x = 5$

c) $R(x) = \frac{x}{x^3+10x^2+9x} = \frac{x}{x(x^2+10x+9)}$

$= \frac{x}{x(x+9)(x+1)} = \frac{1}{(x+9)(x+1)}$

$(x+9)(x+1) = 0$

$x = -9, x = -1$

d) $R(x) = \frac{x^2}{x^2+4}$ ← prime

$x^2 + 4 = 0$

$x^2 = -4$

$\sqrt{x^2} = \sqrt{-4}$

$x = \pm 2i$ = not real

no vertical asymptotes

Find the Horizontal or Oblique Asymptotes of a Rational Function

- If a rational function $R(x)$ is ***proper***, that is *if the degree of the numerator is less than the degree of the denominator*, then the line $y = 0$ is a horizontal asymptote. highest power in denominator
- If a rational function $R(x) = \frac{p(x)}{q(x)}$ is ***improper***, that is, if the *degree of the numerator is greater than or equal to the degree of the denominator*, we can use long division to write the rational function as the sum of a polynomial $f(x)$ plus the proper rational function $\frac{r(x)}{q(x)}$. As $x \to -\infty$ or as $x \to \infty$, $\frac{r(x)}{q(x)} \to 0$, and the graph of $R(x)$ approaches the graph of $f(x)$.
  - The possibilities include:
    1. If $f(x) = b$, a constant, the line $y = b$ is a horizontal asymptote of the graph of R.
    2. If $f(x) = ax + b,\ a \neq 0$, the line $y = ax + b$ is an oblique (slanted) asymptote of the graph of R.
    3. In all other cases, the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

Examples: Find the horizontal or oblique asymptotes, if any, of the graph of the function.

a) $R(x)=\dfrac{x+5}{x^2-3x-10}$ deg 1 / deg 2 — Highest degree in denominator

Horizontal Asymptote
$y=0$

b) $H(x)=\dfrac{6x^2-2x+5}{3x^2-2}$ deg 2 / deg 2 — Degree is the same

$y=\frac{6}{3}$

Horizontal Asymptote
$y=2$

c) $H(x)=\dfrac{8x^3+2x^2-6}{2x^2-3}$ deg 3 / deg 2 — Highest degree in numerator

$$\begin{array}{r} 4x+1 \\ 2x^2-3\overline{)\,8x^3+2x^2+0x-6} \\ -8x^3\quad\quad +12x \\ \hline 2x^2+12x-6 \\ -2x^2\quad\quad +3 \\ \hline 12x-3 \end{array}$$

$H(x)=(4x+1)+\frac{12x-3}{2x^2-3}$

Oblique Asymptote
$y=4x+1$

d) $H(x)=\dfrac{3x^4+4x^2-7}{x^2-2x+5}$ deg 4 / deg 2 — Highest in numerator

$$\begin{array}{r} 3x^2+6x+1 \\ x^2-2x+5\overline{)\,3x^4+0x^3+4x^2+0x-7} \\ -3x^4+6x^3-15x^2 \\ \hline 6x^3-11x^2+0x-7 \\ -6x^3+12x^2-30x \\ \hline x^2-30x-7 \\ -x^2+2x-5 \\ \hline -28x-12 \end{array}$$

$H(x)=3x^2+6x+1+\frac{-28x-12}{x^2-2x+5}$

$y=3x^2+6x+1$ acts as the asymptote (parabolic) not horizontal or oblique

Summary:

Given a rational function $R(x)=\frac{p(x)}{q(x)}$,

1. The **domain** is found by setting the denominator not equal to zero and solving. (Do not simplify first).
2. The ***x*-intercepts** are found by setting the numerator to zero and solving. (Simplify first).
3. The ***y*-intercept** is found by setting $x=0$ and simplifying. (You can do this before or after simplifying the function). Remember, if zero is not in the domain, the function has no *y*-intercept.
4. **Vertical asymptotes** are found by setting the denominator to zero and solving (Simplify first).
5. **Horizontal or Oblique Asymptotes:**
   a) If **proper,** where *degree of numerator < degree of denominator*, the graph has a **horizontal asymptote** at $y=0$.
   b) If **improper,** where *degree of numerator = degree of denominator*, the graph has a **horizontal asymptote** at the line $y=\frac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$.
   c) If **improper,** where *degree of numerator > degree of denominator (by one)*, the graph has an **oblique asymptote** of $y=f(x)$ found by performing long division.
   d) If **improper,** where *degree of numerator > degree of denominator (by more than one)*, then *R* has neither a horizontal nor an oblique asymptote, but the end behavior can be determined using long division.

End Behavior Asymptotes

Horizontal Asymptote:

① Highest power in denominator: $y=0$

② Highest power the same in numerator and denominator: $y=\frac{a}{b}$ * $\frac{a}{b}$ is the ratio of the leading coefficients

$\frac{X}{X^2}$ $\frac{10}{100}$ $\frac{100}{10000}$

.1 .001

Oblique Asymptote: (or other)

③ Highest power in the numerator; use long division to find

Math 1050 – 5.3 Notes

The Graph of a Rational Function

Analyzing the Graph of a Rational Function

1. Factor the numerator and denominator of the rational function and find the **domain** by setting the denominator not equal to zero. (Do this before simplifying).
2. Simplify the rational function.
3. Find any holes in the graph. Set any factors that appeared in both the numerator and denominator of the unsimplified function equal to zero to find the x-coordinate of the hole. Plug the x-coordinate into the simplified function to find the y-coordinate of the hole.
4. Find the ***x*-intercepts** by setting the numerator of the simplified function equal to zero. Find the ***y*-intercept** by plugging in zero for x.
5. Find the **vertical asymptotes** by setting the denominator of the simplified rational function equal to zero.
6. Find the **horizontal asymptotes or oblique asymptotes:**

End Behavior Asypmtote

H.A. a) If the ***degree of the numerator < the degree of the denominator***, then the horizontal asymptote is $y = 0$.

H.A. b) If the ***degree of the numerator = the degree of the denominator***, then the horizontal asymptote is $y = \dfrac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}$.

c) If the ***degree of the numerator > the degree of the denominator***, then the graph has an oblique asymptote (if the degree of the numerator is one more than the degree of the denominator), or the graph approaches a quadratic or higher degree function (if the degree of the numerator is more than one higher than the degree of the denominator). To determine the end behavior, use long division.

7. Use the x-intercepts and vertical asymptotes as **critical points** to divide the graph into intervals and determine where it is above the x-axis (y is positive) and where it is below the x-axis (y is negative). Do this by choosing a number x in each interval and evaluating R there.
8. Analyze the behavior of the graph near each asymptote.
9. Put together all of the information to graph the function.

Example: Analyze the Graph of the Rational Function $R(x) = \dfrac{x+2}{x^2-9}$. $= \dfrac{x+2}{(x-3)(x+3)}$

Domain: $(x-3)(x+3) \neq 0$

$\{x \mid x \neq 3, -3\}$

V.A.: $(x+3)(x-3) = 0$

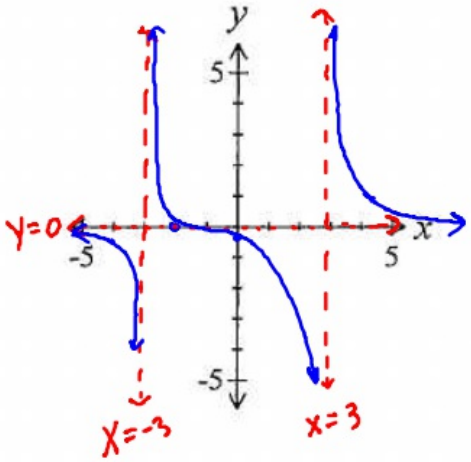
$x = -3,\ x = 3$

Y-int: $R(0) = \dfrac{0+2}{0^2-9} = -\dfrac{2}{9}$ $\left(0, -\dfrac{2}{9}\right)$

X-int: $x+2 = 0$, $x = -2$ $(-2, 0)$

H.A.: highest power in denominator

$y = 0$



x	y
-4	$\frac{-2}{7} = -.3$
-2.5	$\frac{-.5}{-2.8} = .2$
-1	$\frac{1}{-8} = .1$
1	$\frac{3}{-8} = -.4$
4	$\frac{6}{7} = .9$

Example: Analyze the Graph of the Rational Function $R(x)=\frac{x^3+1}{x^2+2x}$. $= \frac{(x+1)(x^2-x+1)}{x(x+2)}$

Domain: $x(x+2)\neq 0$

$\boxed{\{x \mid x\neq 0,-2\}}$

V.A.: $x(x+2)=0$

$\boxed{x=0,\ x=-2}$

y-int: $\boxed{\text{no y-int.}}$ (undefined @ $x=0$)

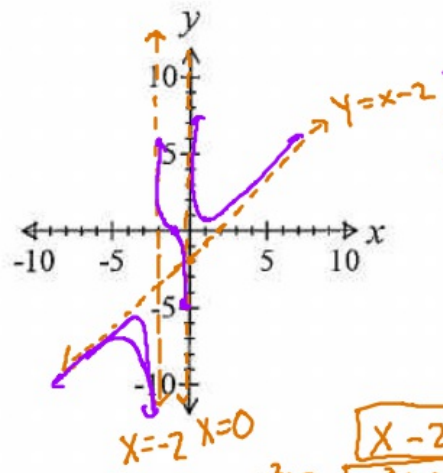
x-int: $(x+1)(x^2-x+1)=0$

$x+1=0$, $x=-1$ $\quad \underbrace{x^2-x+1}_{\text{no x-int}}=0$ $\quad (-1)^2-4(1)(1)$, $1-4=-3$

$\boxed{(-1,0)}$

Highest power in numerator so no H.A.

oblique A.: $\boxed{y=x-2}$



x	y
-1.5	3.2
-.5	-1.2
-3	-8.7
1	.7

$$\begin{array}{r} \boxed{x-2} \\ x^2+2x\,\overline{)\,x^3+0x^2+0x+1} \\ -x^3-2x^2 \\ \hline -2x^2+0x+4 \\ +2x^2+4x \\ \hline 4x+4 \end{array}$$

Example: Analyze the Graph of the Rational Function $R(x)=\frac{x^4+x^2+1}{x^2-1}$. $= \frac{x^4+x^2+1}{(x+1)(x-1)}$

Domain: $(x+1)(x-1)\neq 0$

$\boxed{\{x \mid x\neq -1,1\}}$

V.A: $\boxed{x=-1,\ x=1}$

y-int: $R(0)=\frac{0^4+0^2+1}{0^2-1}=-1$ $\quad \boxed{(0,-1)}$

x-int: $x^4+x^2+1=0$

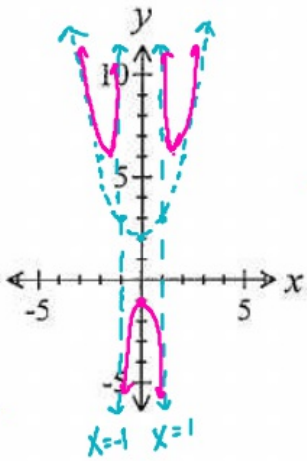
never $=0$ $\quad \boxed{\text{no x-int.}}$

Highest power in numerator

No H.A. or oblique

$\boxed{y=x^2+2}$ acts as an asymptote

x	x^2+2
0	2
±1	3
±2	6



x	y
-3	11.4
-2	7
-1.5	6.7
-.5	-1.8
0	-1
.5	-1.8
1.5	6.7
2	7
3	11.4

$$\begin{array}{r} x^2+2 \\ x^2-1\,\overline{)\,x^4+0x^3+x^2+0x+1} \\ -x^4\quad +x^2 \\ \hline 2x^2+0x+1 \\ -2x^2\quad +2 \\ \hline 3 \end{array}$$

Example: Analyze the Graph of the Rational Function $R(x)=\frac{x^2+x-12}{x^2-4}$ $= \frac{(x+4)(x-3)}{(x+2)(x-2)}$

Domain: $(x+2)(x-2)\neq 0$

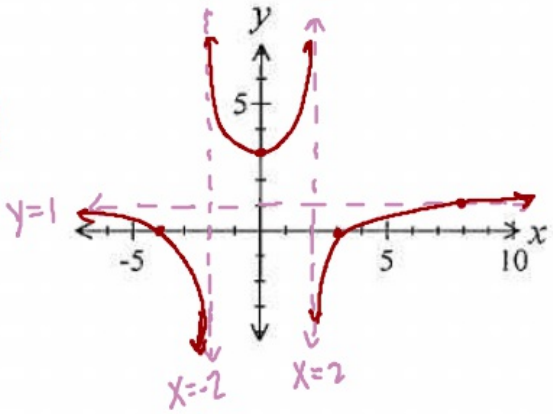
$\{x \mid x \neq -2, 2\}$

V.A. : $x=-2,\ x=2$

y-int: $R(0)=\frac{0^2+0-12}{0^2-4}=3$ $(0,3)$

x-int: $(x+4)(x-3)=0$

$x=-4, 3$ $(-4,0)$ $(3,0)$



Highest Power the same

$y=\frac{1}{1}$; $y=1$

Cross H.A.?

$\frac{x^2+x-12}{x^2-4}=1$

$x^2+x-12=x^2-4$

$+12 \quad +12$

$x=8$ $(8,1)$

Example: Analyze the Graph of the Rational Function $R(x)=\frac{x^2+3x-10}{x^2+8x+15}$ $= \frac{(x-2)(x+5)}{(x+5)(x+3)}$

Ho

Domain: $(x+5)(x+3)=0$

$\{x \mid x \neq -5, -3\}$

Hole @ $x=-5$ $y=\frac{-5-2}{-5+3}=\frac{-7}{-2}=\frac{7}{2}$

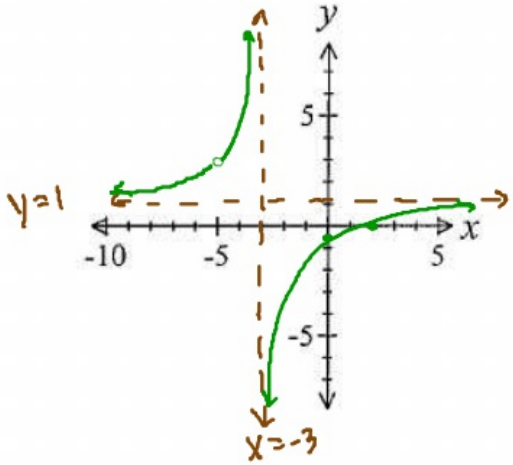
$(-5, \frac{7}{2})$

V.A. : $x=-3$

y-int: $R(0)=\frac{0^2+3(0)-10}{0^2+8(0)+15}=-\frac{2}{3}$ $(0,-\frac{2}{3})$

x-int: $x-2=0$

$x=2$ $(2,0)$



H.A. $y=\frac{1}{1}$ $y=1$

Crosses H.A.? NO

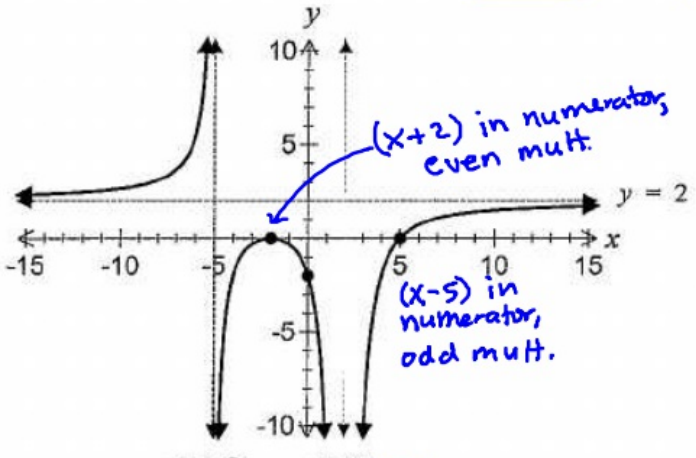
$\frac{x-2}{x+3}=1$

$x-2=x+3$

$-2 \neq 3$

Example: Find a rational function that might have the graph shown below.

★ **Note:** If the graph goes the same direction on both sides of an asymptote (approaches ∞ on both sides or approaches $-\infty$ on both sides), the related factor in the denominator has an even multiplicity. If the graph goes in opposite directions on the two sides of an asymptote (approaches ∞ on one side and $-\infty$ on the other), the related factor in the denominator has an odd multiplicity.



$(x+2)$ in numerator, even mult.

$(x-5)$ in numerator, odd mult.

$(x+5)$ in denominator, odd mult.

$(x-2)$ in denominator, even mult.

Deg. in num. = Deg. in denom.

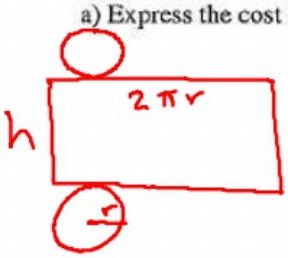
Ratio of leading coefficients is 2

$$R(x) = \frac{2(x+2)^2(x-5)}{(x+5)(x-2)^2}$$

Example: Finding the Least Cost of a Can

Reynolds Metal Company manufactures aluminum cans in the shape of a right circular cylinder with a capacity of 500 cubic centimeters. The top and bottom of the can are made of special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the cans are made of material that costs 0.02¢ per square centimeter.

a) Express the cost C of material for the can as a function of the radius r of the can.



Area of top/bottom: $2\pi r^2$

Area of sides: $2\pi rh$

$C = 2\pi r^2(.05) + 2\pi rh(.02)$

$V = \pi r^2 h$

$\frac{500}{\pi r^2} = \frac{\pi r^2 h}{\pi r^2}$

$\frac{500}{\pi r^2} = h$

$C(r) = 2\pi r^2(.05) + 2\pi r\left(\frac{500}{\pi r^2}\right)(.02)$

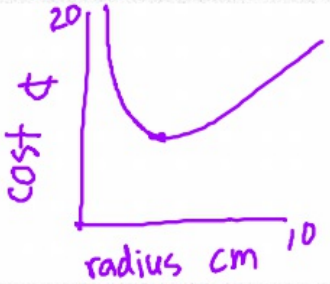
$C(r) = .1\pi r^2 + \frac{20}{r}$ OR $C(r) = \frac{.1\pi r^3 + 20}{r}$

b) Find any vertical asymptotes. Discuss the cost of the can near any vertical asymptotes.

$r = 0$ is a vertical asymptote.

As r gets close to zero, the cost increases.

c) Use a graphing calculator to graph the function $C = C(r)$.



d) What value of r will result in the least cost? What is the least cost?

The cost is lowest when the radius is 3.17cm.

The least cost is 9.47¢ per can.

Math 1050 – 5.4 Notes

Polynomial and Rational Inequalities

Steps for Solving Polynomial and Rational Inequalities

1. Write the inequality so that a polynomial or rational expression f is on the left side and zero is on the right side in one of the following forms:

   $f(x)>0$ $\quad$ $f(x)<0$ $\quad$ $f(x)\geq 0$ $\quad$ $f(x)\leq 0$

   For rational expressions, be sure that the left side is written as a *single quotient* and find its domain.
2. Determine the real numbers at which the expression f on the left side equals zero and, if the expression is rational, the real numbers at which the expression f on the left side is undefined. (In other words, find any x-intercepts and vertical asymptotes of the graph of $y=f(x)$.
3. Use the numbers found in Step 2 to separate the x-axis into intervals.
4. Determine whether the graph of $y=f(x)$ is above or below the x-axis in each interval.

   The solutions of $f(x)>0$ are the x-values for which the graph is **above** the x-axis.

   The solutions of $f(x)\geq 0$ are the x-values for which the graph is **on or above** the x-axis.

   The solutions of $f(x)<0$ are the x-values for which the graph is **below** the x-axis.

   The solutions of $f(x)\leq 0$ are the x-values for which the graph is **on or below** the x-axis.

   ★ **Warning:** For $f(x)\geq 0$ and $f(x)\leq 0$, include the x-intercepts *but not the asymptotes* in the solution set.

-OR-

4. Select a number in each interval and evaluate $f(x)$ at the number.
   a. If the value of $f(x)$ is positive, then $f(x)>0$ for all numbers x in the interval.
   b. If the value of $f(x)$ is negative, then $f(x)<0$ for all numbers x in the interval.

   The solution set is all the intervals that make the inequality true.

   ★ **Warning:** For $f(x)\geq 0$ and $f(x)\leq 0$, include the values for which $f(x)=0$ (the x-intercepts) *but not the values for which* $f(x)$ *is undefined* (the asymptotes) in the solution set.

Example: Solve the inequality $x^4>9x^2$, and graph the solution set.

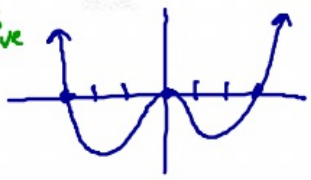
$x^4-9x^2>0$ means positive

x-int: $x^2(x^2-9)>0$

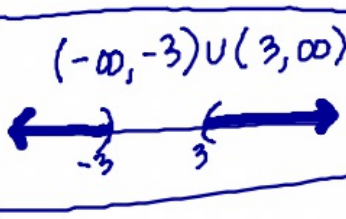
$x^2(x-3)(x+3)>0$

$x=0, 3, -3$

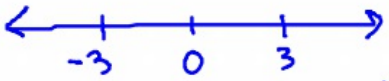
touches, crosses



$(-\infty,-3)\cup(3,\infty)$



OR



intervals	$(-\infty,-3)$	$(-3,0)$	$(0,3)$	$(3,\infty)$
number picked	-4	-1	1	4
$f(\#)$	$f(-4)=112$	$f(-1)=-8$	$f(1)=-8$	$f(4)=112$
pos/neg	pos	neg	neg	pos

end behavior

Example: Solve the inequality $x^4 \geq 8x$, and graph the solution set.

$x^4 - 8x \geq 0$

x-int: $x(x^3 - 8) \geq 0$

$x(x-2)(x^2+2x+4) \geq 0$

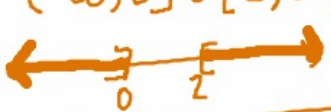
no real roots $b^2 - 4ac < 0$

$x = 0, 2$

crosses

end behavior

$(-\infty, 0] \cup [2, \infty)$



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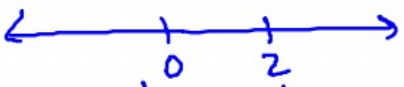


intervals	$(-\infty, 0]$	$[0, 2]$	$[2, \infty)$
pick #	-1	1	3
$f(\#)$	$f(-1) = 9$	$f(1) = -7$	$f(3) = 57$
pos/neg	pos	neg	pos

Example: Solve the inequality $\frac{(x+2)(4-x)}{(x+1)^2} < 0$, and graph the solution set.

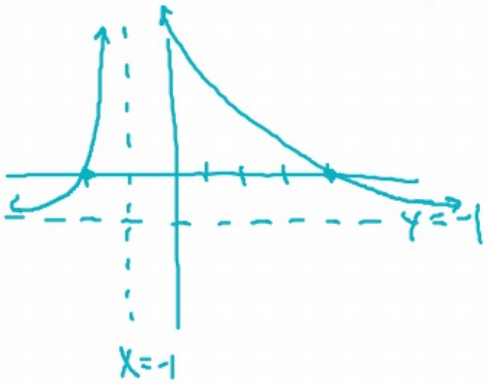
x-int: $(x+2)(4-x) = 0$

$x = -2, 4$

V.A.: $(x+1)^2 = 0$

$x = -1$

H.A. $y = \frac{-x^2}{x^2} = -1$



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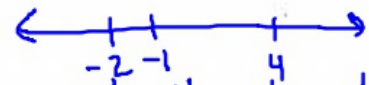


intervals	$(-\infty, -2)$	$(-2, -1)$	$(-1, 4)$	$(4, \infty)$
pick #	-3	-1.5	2	5
$f(\#)$	$f(-3) = \frac{-7}{4}$	$f(-1.5) = 11$	$f(2) = \frac{8}{9}$	$f(5) = \frac{-7}{36}$
pos/neg	neg	pos	pos	neg

$(-\infty, -2) \cup (4, \infty)$

Example: Solve the inequality $\frac{3x+5}{x+4} \leq 2$, and graph the solution set.

$$\frac{3x+5}{x+4} - 2 \leq 0$$

$$\frac{3x+5}{x+4} + \frac{-2(x+4)}{x+4} \leq 0$$

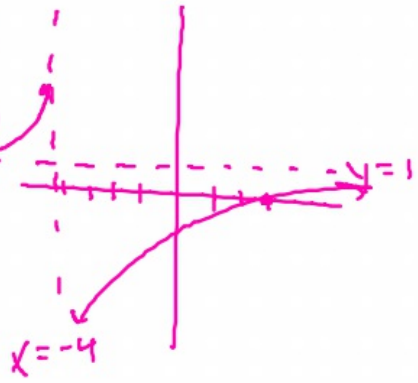
$$\frac{3x+5-2x-8}{x+4} \leq 0$$

$$\frac{x-3}{x+4} \leq 0$$

x-int: $x-3=0$, $x=3$

V.A.: $x+4=0$, $x=-4$

H.A.: $y=1$



OR

intervals	$(-\infty, -4)$	$(-4, 3]$	$[3, \infty)$
pick #	-5	0	4
f(#)	$f(-5) = 8$	$f(0) = \frac{-3}{4}$	$f(4) = \frac{1}{8}$
pos/neg	pos	neg	pos

$(-4, 3]$

Math 1050 – R.6 Notes

Synthetic Division

To find a quotient as well as the remainder when a polynomial is divided by $x-c$, a shortened version of long division, called synthetic division, makes the task simpler.

Steps:

1. Write the constant c of the divisor $x-c$ in a box. **(Opposite sign!)**
2. Write the coefficients of the dividend (including 0's for any missing powers of x) in descending order to the right of the box.
3. Bring the leading coefficient down two rows and enter it into row 3.
4. Multiply the latest entry in row 3 by c, then write the answer in row 2 below the next coefficient in the dividend (one column to the right).
5. Add the entry just written in row 2 to the coefficient above it in row 1. Record the sum in row 3.
6. Repeat steps 4 and 5 until no more entries remain in row 1.
7. The numbers in row 3 are the coefficients of the quotient, with the final number at the right being the remainder.

Example: Solve $(2x^3-x^2+3)\div(x-3)$ using both long division and synthetic division.

Long Division

$$2x^2+5x+15+\frac{48}{x-3}$$

$$x-3\overline{)\,2x^3-x^2+0x+3}$$

$-2x^3+6x^2$

$5x^2+0x+3$

$-5x^2+15x$

$15x+3$

$-15x+45$

48

Synthetic Division

$$\begin{array}{r|rrrr} 3 & 2 & -1 & 0 & 3 \\ & & 6 & 15 & 45 \\ \hline & 2 & 5 & 15 & 48 \end{array}$$

$$2x^2+5x+15+\frac{48}{x-3}$$

Example: Use synthetic division to find the quotient and remainder for the following problem:

$(-4x^3+2x^2-x+1)\div(x+2)$

$$\begin{array}{r|rrrr} -2 & -4 & 2 & -1 & 1 \\ & & 8 & -20 & 42 \\ \hline & -4 & 10 & -21 & 43 \end{array}$$

$-4x^2+10x-21$; R 43

Example: Use synthetic division to show that $x-9$ is a factor of $2x^3-18x^2+x-9$.

If something is a factor, that means it divides evenly with a remainder of zero.

$$\begin{array}{r|rrrr} 9 & 2 & -18 & 1 & -9 \\ & & 18 & 0 & 9 \\ \hline & 2 & 0 & 1 & 0 \end{array}$$

← Remainder is 0, so $x-9$ is a factor

Math 1050 – 5.5 Notes

The Real Zeros of a Polynomial Function

Division Algorithm for Polynomials:

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \text{ or } f(x) = q(x)g(x) + r(x).$$

$f(x)$ is the ***dividend***, $g(x)$ is the ***divisor***, $q(x)$ is the ***quotient***, and $r(x)$ is the ***remainder***.

Remainder Theorem: If a polynomial function $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.

Example: Find the remainder if $f(x) = 2x^3 - 3x^2 + 1$ is divided by

a) $x+1$ $f(-1) = 2(-1)^3 - 3(-1)^2 + 1$
$= -2 - 3 + 1$
$= \boxed{-4}$

b) $x-3$ $f(3) = 2(3)^3 - 3(3)^2 + 1$
$= 2(27) - 3(9) + 1$
$= 54 - 27 + 1$
$= \boxed{28}$

Factor Theorem: If f is a polynomial function, then $x-c$ is a factor of $f(x)$ if and only if $f(c) = 0$. The Factor Theorem means that:

a) If $f(c) = 0$, then $x-c$ is a factor of $f(x)$.

b) If $x-c$ is a factor of $f(x)$, then $f(c) = 0$.

Example: Use the factor theorem to determine whether the function $f(x) = 4x^4 - 15x^2 - 4$ has the factor

a) $x-2$ $f(2) = 4(2)^4 - 15(2)^2 - 4$
$= 64 - 60 - 4$
$= 0$
Factor

b) $x+1$ $f(-1) = 4(-1)^4 - 15(-1)^2 - 4$
$= 4 - 15 - 4$
$= -15$
Not a Factor

Number of Real Zeros Theorem: A polynomial function cannot have more real zeros than its degree.

Rational Zeros Theorem: Let f be a polynomial function of degree 1 or higher, where each coefficient is an integer. The possible rational zeros are all numbers $\frac{p}{q}$, where p is a factor of the constant, and q is a factor of the leading coefficient of the function.

Example: List the potential rational zeros of $f(x) = 3x^4 + 4x^3 + 7x^2 + 8x + 2$.

p Factors of 2: $\pm 1, \pm 2$

q Factors of 3: $\pm 1, \pm 3$

Potential rational zeros: $\boxed{\pm 1, \pm \frac{1}{3}, \pm 2, \pm \frac{2}{3}}$

Example: Find the rational zeros of $f(x) = 3x^4 + 4x^3 + 7x^2 + 8x + 2$.

Test: $f(1) = 3(1)^4 + 4(1)^3 + 7(1)^2 + 8(1) + 2 = 24$

All positive possibilities would give positive values, so they cannot be zeros.

$f(-1) = 3(-1)^4 + 4(-1)^3 + 7(-1)^2 + 8(-1) + 2 = 0$

$f(-\frac{1}{3}) = 3(-\frac{1}{3})^4 + 4(-\frac{1}{3})^3 + 7(-\frac{1}{3})^2 + 8(-\frac{1}{3}) + 2 =$
$\frac{3}{81} + -\frac{4}{27} + \frac{7}{9} - \frac{8}{3} + 2 = 0$

$f(-2) = 3(-2)^4 + 4(-2)^3 + 7(-2)^2 + 8(-2) + 2 = 30$

$f(-\frac{2}{3}) = 3(-\frac{2}{3})^4 + 4(-\frac{2}{3})^3 + 7(-\frac{2}{3})^2 + 8(-\frac{2}{3}) + 2 = -\frac{22}{27}$

Rational zeros: $\boxed{-1, -\frac{1}{3}}$

Theorem: Every polynomial function with real coefficients can be uniquely factored over the real numbers into a product of linear factors and/or irreducible (prime) quadratic factors.

Finding the Real Zeros of a Polynomial Function:

1. Use the degree of the polynomial to determine the maximum number of real zeros.
2. Use the Rational Zeros Theorem to identify the possible rational zeros.
3. Use substitution, synthetic division, or long division to test each potential rational zero.
4. Each time a zero is found, use synthetic division to rewrite the function in factored form. Repeat step 3 on the ***depressed function*** (the function that remains after factoring out a linear factor).
5. Once the depressed function is quadratic, you can factor it or use the quadratic formula to find the remaining zeros.

★ Don't forget to use factoring techniques you already know to help you find the zeros.

Example: Find all of the real zeros of $f(x)=2x^4-x^3-5x^2+2x+2$. Use the zeros to factor f over the real numbers.

Max zeros: 4

p Factors of 2: $\pm 1, \pm 2$

q Factors of 2: $\pm 1, \pm 2$

Potential Rational zeros: $\pm 1, \pm \frac{1}{2}, \pm 2$

$f(1)=0$ ✓

$f(-\frac{1}{2})=0$ ✓

$f(x)=(x-1)(x+\frac{1}{2})(2x^2-4)$ factor out 2

$f(x)=2(x-1)(x+\frac{1}{2})(x^2-2)$ $\quad x^2-2=0,\ x^2=2,\ x=\pm\sqrt{2}$

$f(x)=2(x-1)(x+\frac{1}{2})(x+\sqrt{2})(x-\sqrt{2})$

Zeros: $1, -\frac{1}{2}, -\sqrt{2}, \sqrt{2}$

1	2	−1	−5	2	2
		2	1	−4	−2
	2	1	−4	−2	0

$-\frac{1}{2}$	2	1	−4	−2 | 0
		−1	0	2
	2	0	−4	0

depressed polynomial $2x^3+x^2-4x-2$

Example: Find all real solutions of the equation $2x^3-3x^2-3x-5=0$.

Max zeros: 3

p Factors of −5: $\pm 1, \pm 5$

q Factors of 2: $\pm 1, \pm 2$

Potential Rational zeros: $\pm 1, \pm \frac{1}{2}, \pm 5, \pm \frac{5}{2}$

Test $f(1)=-9$

$f(-1)=-7$

5	2	−3	−3	−5
		10	35	160
	2	7	32	155

−5	2	−3	−3	−5
		−10	65	−310
	2	−13	62	−315

$\frac{1}{2}$	2	−3	−3	−5
		1	−1	−2
	2	−2	−4	−7

$-\frac{1}{2}$	2	−3	−3	−5
		−1	2	$\frac{1}{2}$
	2	−4	−1	−4.5

$\frac{5}{2}$	2	−3	−3	−5
		5	5	5
	2	2	2	0

★

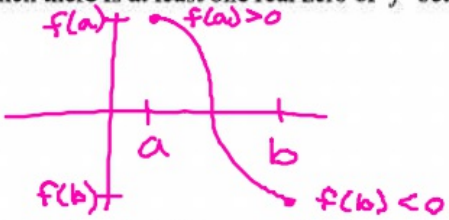
$2x^2+2x+2$

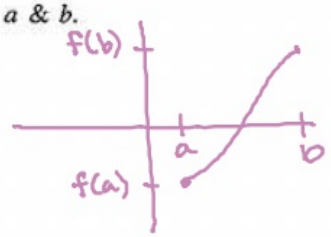
$2(x^2+x+1)$ $\quad b^2-4ac<0$ no real zeros

$f(x)=2(x-\frac{5}{2})(x^2+x+1)$

$x=\frac{5}{2}$ is the only real zero

Intermediate Value Theorem: For polynomial functions if $a < b$, and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one real zero of f between a & b.



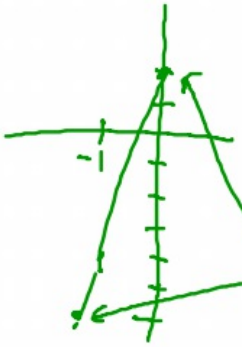


Example: Use the Intermediate Value Theorem to show that $f(x) = x^4 + 8x^3 - x^2 + 2$ has a zero in the interval $[-1,0]$.

$f(-1) = (-1)^4 + 8(-1)^3 - (-1)^2 + 2 = -6$

$f(0) = (0)^4 + 8(0)^3 - (0)^3 + 2 = 2$

> opposite signs, so there is a zero between -1 and 0.



The only way to get from here, to here is to cross the x-axis.

Math 1050 – 5.6 Notes

Complex Zeros; Fundamental Theorem of Algebra

Complex Zero: A complex zero is in the form $a+bi$ or $a-bi$.

Fundamental Theorem of Algebra: Every complex polynomial function $f(x)$ of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form $f(x) = a_n(x-r_1)(x-r_2)...(x-r_n)$, where $a_n, r_1, r_2, ..., r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n complex zeros, some of which may repeat.

Conjugate Pairs Theorem: For any complex polynomial function, if $r = a+bi$ is a zero of f, the complex conjugate $\overline{r} = a-bi$ is also a zero of f. ★ Imaginary solutions always come in pairs.

Corollary: A polynomial f of odd degree with real coefficients has at least one real zero.

Example: A polynomial f of degree 5 has the zeros 2, $3i$, and $4+i$. Find the other two zeros.

$-3i$ $4-i$

(conjugates)

Example: Find a polynomial f of degree 4 that has the zeros 5, –3, and $-2+i$. Additional zero: $-2-i$

$f(x) = (x-5)(x+3)(x-(-2+i))(x-(-2-i))$

$f(x) = (x^2-2x-15)(x+2-i)(x+2+i)$

$f(x) = (x^2-2x-15)(x^2+4x+5)$

$f(x) = x^4+2x^3-18x^2-70x-75$

	x	2	$-i$
x	x^2	$2x$	$-ix$
2	$2x$	4	$-2i$
i	ix	$2i$	$-i^2=1$

$(x+2-i)(x+2+i) = x^2+4x+5$

	x^2	$-2x$	-15
x^2	x^4	$-2x^3$	$-15x^2$
$4x$	$4x^3$	$-8x^2$	$-60x$
5	$5x^2$	$-10x$	-75

Example: Find a polynomial f of degree 5 with the following zeros: 1, i, $4-2i$. Other zeros: $-i$, $4+2i$

$f(x) = (x-1)(x-i)(x+i)(x-(4-2i))(x-(4+2i))$

$f(x) = (x-1)(x^2+ix-ix-i^2)(x-4+2i)(x-4-2i)$

$f(x) = (x-1)(x^2+1)(x-4+2i)(x-4-2i)$

$f(x) = (x^3-x^2+x-1)(x^2-8x+20)$

$f(x) = x^5-9x^4+29x^3-29x^2+28x-20$

	x	-4	$2i$
x	x^2	$-4x$	$2ix$
-4	$-4x$	$+16$	$-8i$
$-2i$	$-2ix$	$8i$	$-4i^2=4$

	x^3	$-x^2$	x	-1
x^2	x^5	$-x^4$	x^3	$-x^2$
$-8x$	$-8x^4$	$8x^3$	$-8x^2$	$8x$
20	$20x^3$	$-20x^2$	$20x$	-20

Example: Find the complex zeros of $f(x)=2x^4+x^3-35x^2-113x+65$. Write f in factored form. 4 zeros

p factors of 65: $\pm 1, \pm 5, \pm 13, \pm 65$

q factors of 2: $\pm 1, \pm 2$

Possible rational zeros: $\pm\left\{1, \frac{1}{2}, 5, \frac{5}{2}, 13, \frac{13}{2}, 65, \frac{65}{2}\right\}$

$f(1)=-80$ $f(-1)=144$ $f(\frac{1}{2})=0$ $f(5)=0$

$\frac{1}{2}$	2	1	−35	−113	65
		1	1	−17	−65
5	2	2	−34	−130	0
		10	60	130	
	2	12	26	0	

$f(x)=(x-\frac{1}{2})(x-5)(2x^2+12x+26)$

$f(x)=2(x-\frac{1}{2})(x-5)(x^2+6x+13)$

use Quadratic Formula

$x=\frac{-6\pm\sqrt{36-52}}{2}=\frac{-6\pm\sqrt{-16}}{2}=\frac{-6\pm 4i}{2}$

$=-3\pm 2i$

Zeros: $\frac{1}{2}, 5, -3+2i, -3-2i$ $f(x)=2(x-\frac{1}{2})(x-5)(x-(-3+2i))(x-(-3-2i))$

$f(x)=2(x-\frac{1}{2})(x-5)(x+3-2i)(x+3+2i)$

Example: Find the complex zeros of $f(x)=x^4+3x^3-19x^2+27x-252$. Write f in factored form.

4 zeros $q=1$

Factors of −252/Potential zeros:

$\pm\{1, 2, 3, 4, 6, 7, 9, 12, 14, 18, 21, 28, 36, 42, 63, 84, 126, 252\}$

1	252
2	126
3	84
4	63
6	42
7	36
9	28
12	21
14	18

$f(-1)=-300$ $f(-2)=-390$ $f(-3)=-504$

$f(-4)=-600$ $f(-6)=-450$ $f(-7)=0$ $f(4)=0$

−7	1	3	−19	27	−252
		−7	28	−63	252
4	1	−4	9	−36	0
		4	0	36	
	1	0	9	0	

$f(x)=(x+7)(x-4)(x^2+9)$

$x^2+9=0$

$\sqrt{x^2}=\sqrt{-9}$

$x=\pm 3i$

Zeros: $-7, 4, 3i, -3i$ $f(x)=(x+7)(x-4)(x-3i)(x+3i)$

Example: One of the zeros of $f(x)=x^4-2x^3-3x^2+10x-10$ is $1+i$. Find the remaining zeros.

Additional zero: $1-i$

$(x-(1+i))$ & $(x-(1-i))$ are factors

$(x-1-i)(x-1+i) = x^2-2x+2$

↑ factor

	x	-1	$-i$
x	x^2	$-x$	$-ix$
-1	$-x$	1	i
i	ix	$-i$	$-i^2=1$

$f(x)=(x^2-2x+2)(x^2-5)$

$f(x)=(x^2-2x+2)(x+\sqrt{5})(x-\sqrt{5})$

$x^2-5=0$

$\sqrt{x^2}=\sqrt{5}$

$x=\pm\sqrt{5}$

$$\begin{array}{r} x^2-5 \\ x^2-2x+2 \overline{)\, x^4-2x^3-3x^2+10x-10} \\ -x^4+2x^3-2x^2 \\ \hline -5x^2+10x-10 \\ +5x^2-10x+10 \\ \hline \end{array}$$

factor

zeros: $1+i,\ 1-i,\ \sqrt{5},\ -\sqrt{5}$