Section 1.8 Combinations of Functions: Composite Functions

Sums, Differences, Products, and Quotients of Two Functions

The **sum** $f+g$ is defined by $(f+g)(x)=f(x)+g(x)$

The **difference** $f-g$ is defined by $(f-g)(x)=f(x)-g(x)$

The **product** $f\cdot g$ is defined by $(f\cdot g)(x)=f(x)\cdot g(x)$

The **quotient** $\frac{f}{g}$ is defined by $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}, g(x)\neq 0$

The domain of $f+g$, $f-g$, or $f\cdot g$ consists of all the numbers x that are in the domains of both f and g. The domain of f/g consists of all the numbers x for which $g(x)\neq 0$ that are in the domains of both f and g.

Examples:

Given $f(x)=x^2$ and $g(x)=1-x$, find $(f+g)(x)$. Then evaluate the sum when $x=2$.

$f(x)+g(x)$

$(x^2)+(1-x)$

$=\boxed{x^2-x+1}$

$(f+g)(2)=(2)^2-(2)+1$

$=4-2+1$

$=\boxed{3}$

Given $f(x)=x^2$ and $g(x)=1-x$, find $(f-g)(x)$. Then evaluate the difference when $x=3$.

$f(x)-g(x)$

$=(x^2)-(1-x)$

$=x^2-1+x$

$=\boxed{x^2+x-1}$

$(f-g)(3)=(3)^2+(3)-1$

$=9+3-1$

$=\boxed{11}$

Given $f(x)=x^2$ and $g(x)=1-x$, find $(fg)(x)$. Then evaluate the product when $x=3$.

$f(x)\cdot g(x)$

$=(x^2)(1-x)$

$=x^2-x^3$

$\boxed{-x^3+x^2}$

$(f\cdot g)(3)=-(3)^3+(3)^2$

$=-27+9$

$=\boxed{-18}$

Find $(f/g)(x)$ and $(g/f)(x)$ for the functions $f(x)=\sqrt{x-3}$ and $g(x)=\sqrt{16-x^2}$. Then find the domains of f/g and g/f.

$(f/g)(x)=\frac{f(x)}{g(x)}=\boxed{\frac{\sqrt{x-3}}{\sqrt{16-x^2}}}$

$\boxed{\text{Domain: } [3,4)}$

$\sqrt{x-3}\rightarrow x-3\geq 0$

$+3 \quad +3$

$x\geq 3$

$\sqrt{16-x^2}: 16-x^2>0$

$(4+x)(4-x)>0$

$-4 \quad 4$

$- \quad + \quad -$

$-4 \quad 4$

test -5

$(4+5)(4--5)$

$- \cdot +$

neg

test 0

$(4+0)(4-0)$

$+ \cdot +$

pos

test 5

$(4+5)(4-5)$

$+ \cdot -$

neg

$-4 \quad 0 \quad 3 \quad 4$

$(g/f)(x)=\frac{g(x)}{f(x)}=\boxed{\frac{\sqrt{16-x^2}}{\sqrt{x-3}}}$

$\boxed{\text{Domain: } (3,4]}$

$16-x^2\geq 0$

$-4 \quad 0 \quad 3 \quad 4$

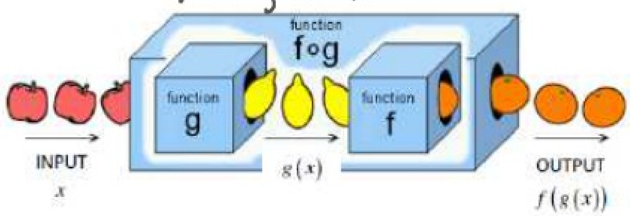
$x-3>0$

$x>3$

Composite Functions

Composite Function: In a composite function, one function is performed, and then a second function is performed on the result of the first function. $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$.

f of g of x



Hints:

- Work inside out. Plug the input into the inside function, then plug the result into the outside function.
- $(f \circ g)(x) = f(g(x))$ is not the same as $(f \cdot g)(x) = f(x) \cdot g(x)$.

  ↑ Composition of functions   ↑ Multiplication of functions

Example: Evaluate each expression using the values given in the table.

x	-3	-2	-1	0	1	2	3
$f(x)$	-7	-5	-3	-1	3	5	7
$g(x)$	8	3	0	-1	0	3	8

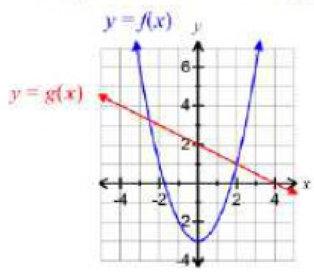
a) $(f \circ g)(-2) = f(g(-2))$
$g(-2) = 3$ $\quad f(3) = \boxed{7}$

b) $(g \circ f)(-1) = g(f(-1))$
$f(-1) = -3$ $\quad g(-3) = \boxed{8}$

c) $(f \circ f)(1) = f(f(1))$
$f(1) = 3$ $\quad f(3) = \boxed{7}$

d) $(g \circ g)(0) = g(g(0))$
$g(0) = -1$ $\quad g(-1) = \boxed{0}$

Example: Evaluate each expression using the graph.



a) $(f \circ g)(4)$
$f(g(4))$
$g(4) = 0$ $\quad f(0) = \boxed{-3}$

b) $(g \circ f)(-1)$
$g(f(-1))$
$f(-1) = -2$ $\quad g(-2) = \boxed{3}$

c) $(f \circ f)(1)$
$f(f(1))$
$f(1) = -2$ $\quad f(-2) = \boxed{1}$

d) $(g \circ g)(0)$
$g(g(0))$
$g(0) = 2$ $\quad g(2) = \boxed{1}$

Example: $f(x) = 2x^2$ and $g(x) = 1 - 3x^2$

a) Find $(f \circ g)(4)$

$f(g(4))$

$g(4) = 1 - 3(4)^2$
$= 1 - 3(16)$
$= 1 - 48$
$= -47$

$f(-47) = 2(-47)^2$
$= 2(2209)$
$= \boxed{4418}$

b) Find $(g \circ f)(2)$

$g(f(2))$

$f(2) = 2(2)^2$
$= 2 \cdot 4$
$= 8$

$g(8) = 1 - 3(8)^2$
$= 1 - 3 \cdot 64$
$= 1 - 192$
$= \boxed{-191}$

c) Find $(f \circ f)(1)$

$f(f(1))$

$f(1) = 2(1)^2$
$= 2 \cdot 1$
$= 2$

$f(2) = 2(2)^2$
$= 2 \cdot 4$
$= \boxed{8}$

d) Find $(g \circ g)(0)$

$g(g(0))$

$g(0) = 1 - 3(0)^2$
$= 1 - 3 \cdot 0$
$= 1 - 0$
$= 1$

$g(1) = 1 - 3(1)^2$
$= 1 - 3 \cdot 1$
$= 1 - 3$
$= \boxed{-2}$

Example: Given $f(x) = 2x + 5$ and $g(x) = 4x^2 + 1$, find the following.

a. $(f \circ g)(x)$

$f(g(x))$

$f(4x^2+1) = 2(4x^2+1)+5$

$= 8x^2+2+5$

$= \boxed{8x^2+7}$

b. $(g \circ f)(x)$

$g(f(x))$

$g(2x+5) = 4(2x+5)^2+1$

$= 4(4x^2+20x+25)+1$

$= 16x^2+80x+100+1$

$= \boxed{16x^2+80x+101}$

c. $(f \circ g)(-\frac{1}{2})$

$f(g(-\frac{1}{2}))$

use $8x^2+7$ OR do the other way

$8(-\frac{1}{2})^2+7$

$= 8(\frac{1}{4})+7$

$= 2+7$

$= \boxed{9}$

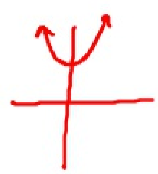
Example: Find the domain of $(f \circ g)(x)$ for the functions $f(x) = \sqrt{x}$ and $g(x) = x^2 + 4$.

$f(g(x))$

$f(x^2+4) = \sqrt{x^2+4}$

Domain:

$x^2+4 \geq 0$

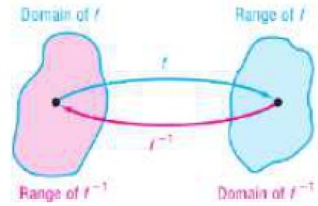


$\boxed{(-\infty, \infty)}$

Section 1.9 Inverse Functions

f^{-1}

Inverse Function: Two functions are ***inverses*** if and only if whenever one function contains the element (a,b), the other function contains the element (b,a). If f is a one-to-one function, the correspondence from the range of f back to the domain of f is called the ***inverse function*** of f. The inverse of f is abbreviated f^{-1}.



★ **Domain of f = Range of f^{-1}** **Range of f = Domain of f^{-1}**

Example: Find the inverse of the following one-to-one function: $\{(2,3),(4,5),(6,8),(9,10),(12,14)\}$

$\{(3,2),(5,4),(8,6),(10,9),(14,12)\}$

If we start with x, apply f, and then apply f^{-1}, we get x back again.
If we start with x, apply f^{-1}, and then apply f, we get x back again.
What f does, f^{-1} undoes, and vice versa. In other words,

$f^{-1}(f(x)) = x$, where x is in the domain of f.

$f(f^{-1}(x)) = x$ where x is in the domain of f^{-1}.

To verify that two functions are inverses, show that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$

Example: Verify that the inverse of $f(x) = \dfrac{2}{x+5}$ is $f^{-1}(x) = \dfrac{2}{x} - 5$.

$f^{-1}(f(x))$

$f^{-1}\left(\frac{2}{x+5}\right) = \frac{2}{\frac{2}{x+5}} - 5$

$= \cancel{2}\cdot\frac{x+5}{\cancel{2}} - 5$

$= x+5-5$

$= x$ ✓

$f(f^{-1}(x))$

$f\left(\frac{2}{x} - 5\right) = \frac{2}{\left(\frac{2}{x} - 5\right) + 5}$

$= \frac{2}{\frac{2}{x}}$

$= \cancel{2}\cdot\frac{x}{\cancel{2}} = x$ ✓

Example: Verify that the inverse of $f(x) = \sqrt[3]{2x}$ is $f^{-1}(x) = \dfrac{x^3}{2}$.

$f^{-1}(f(x))$

$f^{-1}(\sqrt[3]{2x}) = \frac{(\sqrt[3]{2x})^3}{2}$

$= \frac{2x}{2}$

$= x$ ✓

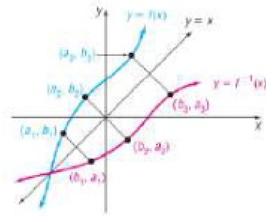
$f(f^{-1}(x))$

$f\left(\frac{x^3}{2}\right) = \sqrt[3]{\cancel{2}\left(\frac{x^3}{\cancel{2}}\right)}$

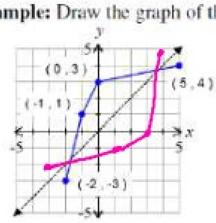
$= \sqrt[3]{x^3}$

$= x$ ✓

Theorem: The graph of a function f and the graph of its inverse f^{-1} are symmetric with respect to the line $y = x$.



Example: Draw the graph of the inverse function.



One-to-one Function: A function is ***one-to-one*** if for any value of x there is exactly one y (otherwise it wouldn't be a function), and for any value of y, there is exactly one x.

Example: Determine whether the following functions are one to one.

a) $\{(-2,6),(-1,3),(0,2),(1,5),(2,8)\}$

yes (y does not repeat)

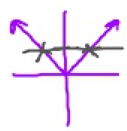
b) $\{(1,1),(2,4),(3,9),(0,0),(-1,1),(-2,4)\}$

not one to one (y values repeat)

Horizontal Line Test: If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

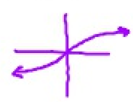
Example: For each function, use its graph to determine whether the function is one-to-one.

a) $f(x) = |x|$



not one to one

b) $g(x) = \sqrt[3]{x}$



yes

Finding the Inverse of a Function

1. Rewrite $f(x)$ as y in the original equation.
2. Interchange x and y.
3. Solve for y.
4. Replace y with the notation $f^{-1}(x)$.

Example: Find the inverse. State the domain and range of $f(x)$ and the domain and range of $f^{-1}(x)$.

a) $f(x) = -3x + 1$

$y = -3x + 1$

$x = -3y + 1$
$-1 \qquad\quad -1$

$\frac{x-1}{-3} = \frac{-3y}{-3}$

$-\frac{1}{3}x + \frac{1}{3} = y$

$\boxed{f^{-1}(x) = -\frac{1}{3}x + \frac{1}{3}}$

$\frac{x-1}{-3} = \frac{-3y}{-3}$

$y = \frac{x-1}{-3}$

f: Domain $(-\infty,\infty)$
Range $(-\infty,\infty)$

f^{-1}: Domain $(-\infty,\infty)$
Range $(-\infty,\infty)$

b) $f(x) = \frac{2x+3}{5x-4}$

$y = \frac{2x+3}{5x-4}$

$(5y-4)\,x = \frac{2y+3}{\cancel{5y-4}}\,\cancel{(5y-4)}$

$5yx - 4x = 2y + 3$
$-2y \qquad\quad -2y$

$5xy - 2y - 4x = 3$
$+4x \qquad +4x$

$5xy - 2y = 4x + 3$

$\frac{y(\cancel{5x-2})}{\cancel{5x-2}} = \frac{4x+3}{5x-2}$

$\boxed{f^{-1}(x) = \frac{4x+3}{5x-2}}$

f: Domain: $(-\infty,\frac{4}{5}) \cup (\frac{4}{5},\infty)$
$5x-4 \neq 0$
$x \neq \frac{4}{5}$
Range: $(-\infty,\frac{2}{5}) \cup (\frac{2}{5},\infty)$

f^{-1}: Domain: $(-\infty,\frac{2}{5}) \cup (\frac{2}{5},\infty)$
$5x-2 \neq 0$
$x \neq \frac{2}{5}$
Range: $(-\infty,\frac{4}{5}) \cup (\frac{4}{5},\infty)$